Stirling's formula

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This states that

$$n! \sim \sqrt{2\pi}e^{-n}n^{n+1/2}.$$

We can divide the proof of Stirling's formula into two parts: first that there is a constant $C > 0$ such that

$$n! \sim Ce^{-n}n^{n+1/2}$$

and second that

$$C = \sqrt{2\pi}.$$

Here I'll give a fairly complete proof of the first part and a very incomplete proof of the second.

Of course $n! = 1 \times 2 \times 3 \times \cdots \times n$ is a product, but sums are easier to deal with than products, so we study

$$\log n! = \log 1 + \log 2 + \log 3 + \cdots + \log n = \log 2 + \log 3 + \cdots + \log n$$

instead. This should approximate the area under the curve $y = \log x$ between $x = 1$ and $x = n$. Indeed applying the trapezium rule with strips of width 1 indicates we should approximate

$$I_n = \int_1^n \log x \, dx$$

by

$$S_n = \frac{\log 1}{2} + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{\log n}{2}.$$

The weak form of Stirling's formula will follow from the analysis of the accuracy of this approximation.

By doing the integral we get

$$I_n = n \log n - n + 1$$

and of course

$$S_n = \log n! - \frac{\log n}{2}.$$

The error bound for the trapezium rule with one strip is

$$\left| \int_a^b f(x)\,dx - (b-a)\frac{f(a)+f(b)}{2} \right| \le \frac{(b-a)^3}{12} \max_{x\in[a,b]} |f''(x)|.$$

Here f is a function on the interval $[a,b]$ with a continuous second derivative. This should be proved in any introductory course on numerical methods, but you can take it as an exercise for the reader. Taking $f(x) = \log x$ and the interval to be $[k, k+1]$ we have $f''(x) = -1/x$ and so

$$\left| \int_k^{k+1} \log x\,dx - \frac{\log k + \log(k+1)}{2} \right| \le \frac{1}{12} \max_{x\in[k,k+1]} \left| \frac{1}{x^2} \right| = \frac{1}{12k^2}.$$

Now

$$I_n - S_n = \sum_{k=1}^{n-1} E_k$$

where

$$E_k = \int_k^{k+1} \log x\,dx - \frac{\log k + \log(k+1)}{2}.$$

Since $|E_k| \le 1/(12k^2)$ and the series $\sum_{k=1}^{\infty} 1/k^2$ is convergent, the series $\sum_{k=1}^{\infty} E_k$ converges absolutely, to L say. Therefore

$$L = \lim_{n\to\infty}(I_n - S_n) = \lim_{n\to\infty}(n\log n - 1 - \log n! + (\log n)/2)$$

and so

$$\lim_{n\to\infty}((n+1/2)\log n - n - \log n!) = L + 1.$$

It follows that

$$\lim_{n\to\infty} \frac{e^{-n} n^{n+1/2}}{n!} = e^{L+1}$$

which is

$$n! \sim Ce^{-n} n^{n+1/2}$$

with $C = e^{-L-1}$.

To prove that $C = \sqrt{2\pi}$ involves some jiggery-pokery, and is the harder part of Stirling's formula. The usual trick involves *Wallis's formula* (which I won't prove):

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \cdots.$$

From Wallis's formula,

$$\begin{aligned}\frac{\pi}{2} &= \lim_{N\to\infty}\prod_{n=1}^{N}\frac{(2n)^2}{(2n-1)(2n+1)} = \lim_{N\to\infty}\prod_{n=1}^{N}\frac{(2n)^4}{(2n-1)(2n)^2(2n+1)} \\ &= \lim_{N\to\infty}\frac{2^{4N}N!^4}{(2N+1)(2N)!^2},\end{aligned}$$

We now apply our weak version of Stirling's formula:

$$\frac{2^{4N}N!^4}{(2N+1)(2N)!^2} \sim \frac{2^{4N}N!^4}{(2N)(2N)!^2} \sim \frac{2^{4N}C^4e^{-4N}N^{4N+2}}{C^2e^{-4N}(2N)^{2N+2}} = \frac{C^2}{4}.$$

Therefore $\pi/2 = C^2/4$, that is $C = \sqrt{2\pi}$.