Combinatorics: Problem sheet 1

Solutions to indicated questions must be submitted by Thursday 29 October 2015
You are encouraged to submit solutions to other questions

If you use outside resources (books, papers, websites etc.) to help with your solutions you should acknowledge and ***cite*** *them carefully*

Recall that, when $n \in \mathbf{N}$, $[n]$ is shorthand for the set $\{1, 2, \ldots, n\}$. For instance $[5] = \{1, 2, 3, 4, 5\}$. Also a k-*subset* of a set A means a k-element subset of A.

1. Consider paths on the square grid. We allow three types of step:

   - R: moving one unit right, from (x, y) to $(x + 1, y)$,
   - D: moving one unit down, from (x, y) to $(x, y - 1)$,
   - K: a "knight's move", from (x, y) to $(x + 1, y + 2)$.

   Using these steps, how many allowable paths are there from $(0, 0)$ to $(m, 0)$ which use exactly t K-steps? [**20 marks**]

   In particular, how many allowable paths in all are there from $(0, 0)$ to $(3, 0)$? [**10 marks**]

2. Recall that the number of solutions of

   $$x_1 + x_2 + \cdots + x_r = n \qquad (*)$$

   with all the $x_i \in \{0, 1, 2, \ldots\}$ is $\binom{n+r-1}{r-1}$.

   How many solutions of $(*)$ are there where all $x_i \in \{1, 2, 3 \ldots\}$?

3. Prove that the number of k-subsets of $\{1, 2, \ldots, n\}$ whose largest element is r (for $k \leq r \leq n$) is $\binom{r-1}{k-1}$. Deduce that

   $$\binom{n}{k} = \sum_{r=k}^{n} \binom{r-1}{k-1}.$$

4. Let $k_1, \ldots, k_r$ be positive integers and $n = k_1 + \cdots + k_r$. Prove that

   $$\binom{n}{k_1, \ldots, k_r} = \binom{n-1}{k_1 - 1, k_2 \ldots, k_r} + \binom{n-1}{k_1, k_2 - 1 \ldots, k_r} + \cdots + \binom{n-1}{k_1, k_2 \ldots, k_r - 1}.$$

5. Consider a pack of $m + n$ cards (all different) with m of the cards coloured red and the other n coloured blue. By counting the number of r-card hands that can be dealt from this pack according to the number of red cards in the hand, prove that

$$\binom{m+n}{r} = \sum_k \binom{m}{k}\binom{n}{r-k}.$$

(Here the summation is over all integers where the right side makes sense: where $0 \leq k \leq m$ and $0 \leq r - k \leq n$.)

6. Let us call a subset S of $[n]$ *unfriendly* if it contains no two consecutive numbers (since no number wants its neighbour in the set!). For instance, the unfriendly subsets of $[5]$ are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$ and $\{1,3,5\}$.

   (a) Find all unfriendly subsets of $[6]$ and of $[7]$. [**5 marks**]

   (b) Let $U(n)$ denote the number of unfriendly subsets of $[n]$. Prove that $U(n+2) = U(n+1) + U(n)$. [**5 marks**]

   (c) Let $U(n,k)$ denote the number of unfriendly k-subsets of $[n]$. Prove that $U(n,0) = 1$, $U(n,1) = n$ and $U(n,2) = \binom{n-1}{2}$. [**10 marks**]

   (d) Conjecture a general formula for $U(n,k)$. Prove it. [**10 marks**]

7. How many anagrams has the word PARALLELOTOPE?

8. For each positive integer a, let $f(a)$ denote its largest odd divisor. We tabulate the first few values of f below.

a	1	2	3	4	5	6	7	8	9	10	11	12
$f(a)$	1	1	3	1	5	3	7	1	9	5	11	3

   Prove that if $a < b$ and $f(a) = f(b)$ then a is a divisor of b. Using this, and the pigeonhole principle, prove the following theorem: if S is an $(n+1)$-subset of $[2n]$ there are distinct numbers a, $b \in S$ such that a is a divisor of b. [**20 marks**]

   Show that, for each positive integer n, there is an n-subset T of $[2n]$ such that there are no distinct a, $b \in T$ with a a divisor of b. [**5 marks**]

9. Let S be a square with side of length 1. Prove that if ten points are chosen inside S, then there are two of these points at distance at most $\sqrt{2}/3$ apart. Also find a set of nine points in S each of which is at distance more than $\sqrt{2}/3$ from all the others.

10. (a) How many integers between 1001 and 2000 inclusive are divisible by none of the numbers 2, 3 and 5.

    (b) How many integers between 5001 and 6000 inclusive are divisible by none of the numbers 2, 3, 5, and 7. [(b) **10 marks**]

11. A positive integer is *squarefree* if it is not divisible by a square of a prime number. For instance, the squarefree numbers up to 10 are 1, 2, 3, 5, 6, 7 and 10. Use the inclusion-exclusion principle to count the number of squarefree numbers between 101 and 200 inclusive.

12. Let D_n denote the number of derangements of $[n]$, that is, the number of permutations of $[n]$ with no fixed points. Now let $D(k,n)$ denote the number of permutations of $[n]$ with exactly k fixed points. For instance, $D(0,n) = D_n$ and $D(n,n) = 1$. Also $D(1,3) = 3$ since this number counts the permutations (1)(2 3), (2)(1 3) and (3)(1 2), and $D(2,3) = 0$.

    Prove that $D(1,n) = nD_{n-1}$ and find a similar formula for $D(k,n)$ in general.

    For each integer $k \geq 0$ find $\lim_{n\to\infty} D(k,n)/n!$.

13. Assuming Stirling's formula, in each case find real numbers α, β and γ with $a_n \sim \alpha n^{\beta}\gamma^n$:

    (a) $a_n = \binom{3n}{n}$; (b) $a_n = \dfrac{(3n)!(2n)!}{(4n)!n!}$ [(b) **5 marks**]; (c) $a_n = \dfrac{1}{n+1}\binom{2n}{n}$.

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