Combinatorics: Problem sheet 2

Solutions to indicated questions must be submitted by Thursday 12 November 2015
You are encouraged to submit solutions to other questions

If you use outside resources (books, papers, websites etc.) to help with your solutions you should acknowledge and ***cite*** *them carefully*

Recall that, when $n \in \mathbf{N}$, $[n]$ is shorthand for the set $\{1, 2, \ldots, n\}$. For instance $[5] = \{1, 2, 3, 4, 5\}$. Also a k-*subset* of a set A means a k-element subset of A.

1. Let U_n denote the number of ways to divide a 2-by-n rectangle into 2-by-1 dominoes.

   Show that in such an arrangement, the number of horizontal dominoes is even. Moreover prove that the number of ways of dividing a 2-by-n rectangle into dominoes with so that $2j$ dominoes are horizontal is $\binom{n-j}{j}$ and deduce that

   $$U_n = \sum_j \binom{n-j}{j}$$

   where this sum is over all the integers j with $0 \le j \le n/2$. [**8 marks**]

   Also prove that

   $$U_{m+n} = U_m U_n + U_{m-1} U_{n-1}$$

   as long as $m \ge 1$ and $n \ge 1$. [**7 marks**]

2. For each integer $n \ge 0$ one has

   $$\frac{1}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} t^k. \qquad (*_n)$$

   Prove $(*_n)$

   (a) directly, using the binomial theorem;

   (b) by induction, multiplying $(*_n)$ by $1/(1-t)$;

   (c) by induction, differentiating $(*_n)$.

3. For each of the following recursively defined sequences (a_n), find its generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$, and hence find a formula for a_n:

   (a) $a_0 = 1$, $a_1 = 0$, $a_n = 10a_{n-1} - 21a_{n-2}$ $(n \ge 2)$;

(b) $a_0 = 2$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$;

(c) $a_0 = 1$, $a_1 = 3$, $a_n = 6a_{n-1} - 10a_{n-2}$ $(n \geq 2)$; **[6 marks]**

(d) $a_0 = 1$, $a_1 = 6$, $a_n = 6a_{n-1} - 9a_{n-2}$ $(n \geq 2)$;

(e) $a_0 = 0$, $a_1 = -1$, $a_2 = 5$, $a_n = 3a_{n-1} - 4a_{n-3}$ $(n \geq 3)$. **[9 marks]**.

4. (From 2014 exam) You have a supply of cards, each coloured red, blue or green. You arrange n of these cards in a row. Such an arrangement is called *admissible* if

   - no two blue cards are adjacent, and
   - no green card has a red or blue card to its right.

   For example $RBRRBGGG$ is an admissible arrangement, but $RBBRBGGG$ and $RBRRBGBG$ are not.

   Let r_n denote the number of admissible n-card arrangements having a red card as the right-most card. Similarly let b_n and g_n respectively denote the numbers of admissible n-card arrangements having a blue or green card respectively as the right-most card. Define

$$R(t) = \sum_{n=1}^{\infty} r_n t^n, \quad B(t) = \sum_{n=1}^{\infty} b_n t^n \quad \text{and} \quad G(t) = \sum_{n=1}^{\infty} g_n t^n.$$

   Prove that

$$R(t) = t + tR(t) + tB(t)$$

   and give similar formulas for $B(t)$ and $G(t)$. Hence find an explicit formula for $G(t)$ and use that to find an explicit formula for g_n. **[30 marks]**

5. Let $a_0, a_1, \ldots$ be a sequence of numbers with generating function $A(t) = \sum_{n=0}^{\infty} a_n t^n$. Define a new sequence $s_0, s_1, \ldots$ by $s_n = a_0 + a_1 + \cdots + a_n = \sum_{j=0}^{n} a_j$. Prove that

$$\sum_{n=0}^{\infty} s_n t^n = \frac{A(t)}{1-t}.$$

   Deduce that

$$\sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{e^{-t}}{1-t}$$

   where D_n denotes the number of derangements of n.

6. Recall that a permutation of $[n]$ is a bijective mapping from $[n]$ to $[n]$.

   Let f be a permutation of $[n]$. We call a number j a *descent* of f if $f(j) > f(j+1)$. For instance the permutation f with $f(1), \ldots, f(6)) =$

$(5, 3, 2, 6, 4, 1)$ has has four descents, namely 1, 2, 4 and 5. (These descents correspond to the positions in the sequence $(5, 3, 2, 6, 4, 1)$ immediately before a decrease.) It's plain that a permutation of $[n]$ must have between 0 and $n - 1$ descents.

Define $E(n, k)$ is the number of permutations of $[n]$ with k descents. Prove that

(a) $\sum_{k=0}^{n-1} E(n, k) = n!$;

(b) $E(n, 0) = E(n, n - 1) = 1$;

(c) $E(n, k) = E(n, n - k - 1)$ for all n and k;

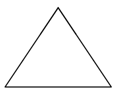
(d) $E(n, k) = (k + 1)E(n - 1, k) + (n - k)E(n - 1, k - 1)$ whenever $0 < k < n - 1$.

Using the last recurrence, compute $E(n, k)$ for all n and k with $1 \leq n \leq 6$ and $0 \leq k \leq n - 1$.

7. Let $(a_1, a_2, \ldots, a_n)$ be a sequence of n nonnegative integers. We say this sequence is *admissible* if $a_1 \geq 1$, $a_1 + a_2 \geq 2$, etc. (that is the sum of the first k terms is at least k, and also $a_1 + a_2 + \cdots + a_n = n$. Let A_n be the number of admissible sequences of length n. Then, for instance, $A_3 = 5$ as $(1, 1, 1)$, $(1, 2, 0)$, $(2, 0, 1)$, $(2, 1, 0)$ and $(3, 0, 0)$ are the admissible sequences of length three.

   Find A_4 (and A_1 and A_2). Conjecture, and prove, a general formula for A_n.

8. Let T_n denote the number of ways of cutting a (convex) polygon with $n + 2$ vertices into n triangles. Then $T_1 = 1$:



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$T_2 = 2$:

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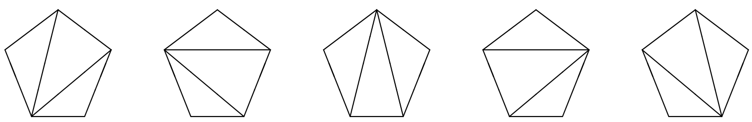
and $T_3 = 5$:



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Find T_4, and conjecture a formula for T_n. **[10 marks]**

9. Recall that a *Dyck path* consists of two types of steps:

   - Up: $(x, y) \to (x + 1, y + 1)$
   - Down: $(x, y) \to (x + 1, y - 1)$

   starts at $(0, 0)$, ends at $(2n, 0)$ for some n and never descends below the x-axis. There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ Dyck paths from $(0, 0)$ to $(2n, 0)$.

   An *almost-Dyck path* from $(0, 0)$ to $(2n, 0)$ consists of Up and Down steps where exactly two of its steps lie below the x-axis. Show that each almost-Dyck path from $(0, 0)$ to $(2n, 0)$ goes through exactly one point $(2k + 1, -1)$, and that there are $C_k C_{n-k-1}$ almost-Dyck paths from $(0, 0)$ to $(2n, 0)$ through this point. Hence find and prove a formula for the total number of almost-Dyck paths from $(0, 0)$ to $(2n, 0)$.

10. Consider paths in the grid. We allow three types of steps:

    - Up: $(x, y) \to (x + 1, y + 1)$;
    - Down: $(x, y) \to (x + 1, y - 1)$;
    - Horizontal: $(x, y) \to (x + 1, y)$.

    An *M-path* is a finite path built from these steps, starting at $(0, 0)$, ending on the x-axis and entirely lying on or above the x-axis. Let M_n denote the number of M-paths ending at $(n, 0)$ (also let $M_0 = 1$). Find M_1, M_2, M_3, M_4 and M_5, directly from the definition. [**12 marks**]

    Prove that, for $n \geq 2$,

    $$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2} M_{n-k}.$$

    Use this identity to find a quadratic equation satisfied by the generating function $M(t) = \sum_{n=0}^{\infty} M_n t^n$ and hence find an explicit expression for $M(t)$. [**18 marks**]

    (Warning: as far as I know there is no "closed formula" for the M_n themselves.)

RJC 8/10/2015