Combinatorics: Problem sheet 3

Solutions to indicated questions must be submitted by Thursday 26 November 2015
You are encouraged to submit solutions to other questions

If you use outside resources (books, papers, websites etc.) to help with your solutions you should acknowledge and ***cite*** *them carefully*

Recall that, when $n \in \mathbf{N}$, $[n]$ is shorthand for the set $\{1, 2, \ldots, n\}$. For instance $[5] = \{1, 2, 3, 4, 5\}$.

1. Calculate the Stirling numbers of the second kind $S(n, k)$ for $1 \leq k \leq n \leq 8$. Hence compute the Bell numbers up to B_8. Check these satisfy the Bell recurrence.

2. The entries in the third column of the Stirling triangle (of the second kind) have the generating function

$$\sum_{n=3}^{\infty} S(n,3)t^n = \frac{t^3}{(1-t)(1-2t)(1-3t)}.$$

Using this, find a formula for $S(n, 3)$.

3. The *Stirling numbers of the first kind* are denoted by $s(n, k)$ and are defined whenever $1 \leq k \leq n$ by

$$s(n,1) = (n-1)!,$$

$$s(n,n) = 1$$

and

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k)$$

for $1 < k < n$.

Calculate the $s(n, k)$ for $1 \leq k \leq n \leq 6$.

Prove that

(a) $s(n, n-1) = \binom{n}{2}$;

(b) $s(n, 2) = (n-1)! \sum_{k=1}^{n-1} 1/k$;

(c) $\sum_{k=1}^{n} x^k s(n, k) = x(x+1)(x+2)\cdots(x+n-1)$; and

(d) $\sum_{k=1}^{n} (-1)^k s(n, k) = 0$ provided that $n \geq 2$.

[20 marks overall]

4. This question considers the generating functions of the "diagonals" of the Stirling triangle.

   For each integer $r \geq 0$, define

$$G_r(t) = \sum_{n=1}^{\infty} S(n+r, n) t^n.$$

   Prove that for $r > 0$

$$G_r(t) = \frac{t}{1-t} G'_{r-1}(t)$$

   and so find $G_0(t)$, $G_1(t)$ and $G_2(t)$.

5. Let k be a positive integer. Define the exponential generating function

$$H_k(t) = \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!}.$$

   Prove that for $k \geq 2$

$$H'_k(t) = H_{k-1}(t) + kH_k(t).$$

   Deduce that

$$H_k(t) = \frac{(e^t - 1)^k}{k!}$$

   for all integers $k \geq 1$. **[20 marks overall]**

6. In the lectures I defined

$$\left(\frac{1 - e^{-x}}{x} \right)^{-1} = \sum_{m=0}^{\infty} c_m x^m$$

   and found $c_0 = 1$, $c_1 = 1/2$, $c_2 = 1/12$, $c_3 = 0$ and $c_4 = -1/720$.

   Prove that if k is odd and $k \geq 3$ then $c_k = 0$.

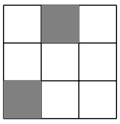
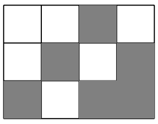
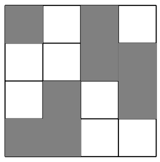
7. In the lectures I calculated

$$\sum_{m=1}^{n} m^4.$$

   Use the same method to calculate

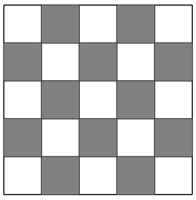
$$\sum_{m=1}^{n} m^5 \quad \textbf{[10 marks]} \qquad \text{and} \qquad \sum_{m=1}^{n} m^6 \quad \textbf{[10 marks]}.$$

8. Calculate the rook polynomials of the (white squares of) the following boards:

(a)  (b)  (c) 

[(c) 10 marks]

9. Let B be the board formed by the white squares of this 5-by-5 chessboard.



Find the rook polynomial of B. **[10 marks]**

10. Download Tim Chow's paper, 'A short proof of the rook reciprocity theorem' (published in Volume 3 of the *The Electronic Journal of Combinatorics:* `http://www.combinatorics.org`).

Chow's rook polynomial $R(B; x)$ is **different** to the rook polynomial $r_B(x)$ I discussed in the lectures. Explain carefully the difference between my definition of $r_B(x)$ and Chow's definition of $R(B; x)$. If a board B contained in the 3-by-3 square has

$$r_B(x) = 1 + 5x + 6x^2 + 2x^3$$

then what is $R(B; x)$?

Calculate $R(B; x)$ for the board B in Q 8(a). Also calculate $r_{\overline{B}}(x)$ and $R(\overline{B}; x)$ for the complementary board $\overline{B}$. Verify Chow's theorem for this choice of B, that is, calculate both sides of the equation in the statement, and check they really are equal. **[20 marks overall]**

RJC 28/10/2015