Combinatorics: Problem sheet 4

Solutions to indicated questions must be submitted by Thursday 10 December 2015
You are encouraged to submit solutions to other questions

If you use outside resources (books, papers, websites etc.) to help with your solutions you should acknowledge and ***cite*** *them carefully*

Recall that p_n denotes the number of partitions of n.

1. Find all partitions of 8, 9 and 10.

2. Find all partitions into distinct parts of 17. Also find all partitions into odd parts of 17.

3. Find all self-conjugate partitions of 21 and draw the Ferrers diagram of each. **[10 marks]**

4. A *composition* of n is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers whose sum is n. Here order does matters: for instance, (5), $(3, 2)$, $(2, 3)$ and $(1, 2, 1, 1)$ are all compositions of 5, and $(3, 2)$ and $(2, 3)$ are **different** compositions. Let c_n denote the number of compositions of n and compute c_1, c_2, c_3 and c_4. Conjecture and prove a formula for c_n. **[10 marks]**

   Why do you think the theory of compositions is less studied than that of partitions?

5. Let a_n be the number of partitions of n in which no part occurs more than twice (e.g., $4\ 3^2\ 2\ 1^2$ is included in this count but $5\ 4\ 1^3$ isn't). Let b_n be the number of partitions of n in which no part is a multiple of 3 (e.g., $7\ 5^2\ 4\ 2^3$ is included but $8\ 6\ 4^3\ 2^2$ isn't). Find infinite product expressions for

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n x^n$$

   and hence, or otherwise, prove that $a_n = b_n$ for all n. **[15 marks]**

6. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and let $\mu = \lambda' = (\mu_1, \ldots, \mu_r)$ be its conjugate. Prove that μ_k is the largest number j with $\lambda_j \geq k$.

7. Let $p_n^{(k)}$ be the number of partitions of n with exactly k parts. Show that

$$\sum_{n=0}^{\infty} p_n^{(k)} t^n = t^k \prod_{j=1}^{k} \frac{1}{1 - t^j}.$$

8. Use Euler's recurrence

$$p_n = \sum_{j=1}^{\infty} (-1)^{j-1} (p_{n-j(3j-1)/2} + p_{n-j(3j+1)/2})$$

to calculate p_n for all n up to 25. **[15 marks]**

(Optional: program a computer to calculate p_n for all n up to 200 (or beyond!) thus checking Major MacMahon's calculations).

9. Prove that $p_n \geq p_{n-1}$ for all $n \geq 1$. Using Euler's recurrence or otherwise, prove that $p_n \leq p_{n-1} + p_{n-2}$ for all $n \geq 2$. **[20 marks]**

10. Recall the *Jacobi triple product*:

$$\prod_{n=1}^{\infty} \Big[(1 + x^{2n-1}z)(1 + x^{2n-1}z^{-1})(1 - x^{2n})\Big] = \sum_{m=-\infty}^{\infty} z^m x^{m^2}. \qquad (*)$$

As an example of the use of $(*)$, setting $z = 1$ and $x = t$ gives

$$\prod_{n=1}^{\infty} \Big[(1 + t^{2n-1})^2 (1 - t^{2n})\Big] = \sum_{m=-\infty}^{\infty} t^{m^2}.$$

(a) Use $(*)$ to obtain a product formula for $\sum_{m=0}^{\infty} t^{m(m+1)}$.

(b) (harder) Put $x = t$ and $z = -tu$ in $(*)$, divide both sides by $(1 - u)$, set $u = 1$ and deduce that

$$\prod_{n=1}^{\infty} (1 - t^{2n})^3 = \sum_{m=0}^{\infty} (-1)^m (2m + 1) t^{m(m+1)}.$$

11. Let $P(t) = \sum_{n=0}^{\infty} p_n t^n = \prod_{m=1}^{\infty} (1 - t^m)^{-1}$. Prove that

$$\frac{P'(t)}{P(t)} = \frac{d}{dt} \log P(t) = \sum_{n=1}^{\infty} \sigma(n) t^{n-1}$$

where $\sigma(n)$ is the sum of the positive integer divisors of n. (For instance, $\sigma(10) = 18$ as the divisors of 10 are 1, 2, 5 and 10 which add to 18). Deduce that

$$np_n = \sum_{k=1}^{n-1} \sigma(k) p_{n-k}.$$

12. Find in the literature (books, papers, websites and so on) a proof of the Jacobi triple product formula that is different to the one I gave in the lectures, and give an account of this in your own words. You should cite your source(s) carefully; marks will be awarded by convincing me that you understand the proof you give. (Warning: there are erroneous proofs out there; correcting errors will impress me, repeating them will depress me.) **[30 marks]**

13. (Exam 2013) Let $\mathcal{A}_k$ denote the set of all partitions which have both (i) exactly k parts and (ii) largest part k. Write down all elements of $\mathcal{A}_3$ and their Ferrers diagrams. State and prove a formula for the number of elements of $\mathcal{A}_k$.

14. (Challenge problem: due to Mircea Merca, *American Mathematical Monthly*, November 2012) We adopt the convention that $p_n = 0$ whenever n is a negative integer. Prove that

$$p_n - 4p_{n-3} + 4p_{n-5} - p_{n-8} > 0$$

whenever $n \geq 0$ and $n \neq 3$.

15. (Challenge problem: due to Mircea Merca, *American Mathematical Monthly*, October 2013) For convenience in displaying the formula below I write $p(n)$ instead of p_n. Prove that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right) = 1$$

for all integers $n \geq 0$.

16. (Challenge problem: more of a mini-project than a straight problem) The first stage in my proof of the Jacobi triple product is to prove that

$$\prod_{k=1}^{\infty}(1 + x^{2k-1}z)(1 + x^{2k-1}z^{-1}) = F(x) \sum_{m=-\infty}^{\infty} x^{m^2} z^m. \qquad (*)$$

Prove that the sum $S(x,z) = \sum_{m=-\infty}^{\infty} x^{m^2} z^m$ satisfies the partial differential equation

$$x\frac{\partial S}{\partial x} = z^2 \frac{\partial^2 S}{\partial z^2} + z\frac{\partial S}{\partial z}. \qquad (\dagger)$$

Use $(*)$ and $(\dagger)$ to find an ordinary differential equation satisfied by $F(x)$ and solve it to complete the proof of the Jacobi triple product.

RJC 16/11/2015