Combinatorics: Problem sheet 5

This is not for assessment, but I'll be happy to mark and comment on any attempts you hand directly to me

1. Recall the construction of a Steiner Triple System of order $3m$ (where m is odd) in the lectures. Its points are $A_0, \dots, A_{m-1}$, $B_0, \dots, B_{m-1}$, $C_0, \dots, C_{m-1}$, and its blocks are $\{A_i, B_i, C_i\}$ and for $i \neq j$, $\{A_i, A_j, B_k\}$, $\{B_i, B_j, C_k\}$ and $\{C_i, C_j, A_k\}$ where $i + j \equiv 2k \pmod{m}$.

   Take $m = 9$ and complete the following 2-sets to blocks: (i) $\{A_2, A_5\}$, (ii) $\{A_3, B_6\}$, (iii) $\{B_7, C_7\}$, (iv) $\{B_0, B_7\}$, (v) $\{A_1, C_8\}$.

2. Let $(X, \mathcal{B})$ be a t-(v, k, λ) design, that is X is a set of v elements and $\mathcal{B}$ is a set of k-element subsets of X obeying the conditions for the blocks of t-(v, k, λ) design. In this question suppose also that $t \geq 2$.

   Let $1 \leq t' < t$. Prove that $(X, \mathcal{B})$ is also a t'-(v, k, λ') design for some integer λ'. Express λ' in terms of t, t', v, k and λ.

3. For the following systems of parameters t-(v, k, λ), determine whether or not a design with those parameters exists (either construct such a design or prove its nonexistence): (i) 2-$(11, 3, 1)$, (ii) 3-$(12, 4, 1)$, (iii) 2-$(57, 8, 1)$.

4. One way of creating designs is to exploit *difference sets*. We let $X = \mathbf{Z}_n = \{0, 1, 2, \dots, n-1\}$ for some integer n. To illustrate the method, take $n = 7$ and consider the seven sets of the form $\{a, a+1, a+3\}$ (with addition modulo 7). In other words we take the set $\{0, 1, 3\}$ and all sets formed by adding the same number to all its elements. These form the blocks of a 2-$(7, 3, 1)$ design. Why is this? Take distinct a, $b \in \mathbf{Z}_7$. If $b - a \equiv 1 \pmod 7$ they lie in the block $\{a, a+1, a+3\}$. If $b - a \equiv 2 \pmod 7$ they lie in the block $\{a-1, a, a+2\}$. If $b - a \equiv 3 \pmod 7$ they lie in the block $\{a, a+1, a+3\}$. If $b - a \equiv 4$, 5 or 6 $\pmod 7$ then $a - b \equiv 3$, 2 or 1 $\pmod 7$ and so we have already done these cases.

   In general we take one or several basic blocks (in the above example $\{0, 1, 3\}$) and add each $a \in \mathbf{Z}_n$ to each element of the basic block. Hopefully (maybe after some trial and error) we get a design. The argument above generalizes: if we have a collection of basic blocks, then for nonzero $r \in \mathbf{Z}_n$ each pair $\{a, a+r\}$ lies in λ_r blocks where λ_r depends on r but not a. If we can arrange it so that all the $\lambda_r = \lambda$ are the same, then we get a 2-(n, k, λ) design.

(a) Let $n = 13$. We aim to construct a 2-$(13, 3, 1)$ design. This must have 26 blocks, so we need two basic blocks. Take one of the basic blocks to be $\{0, 1, 4\}$ (so that 13 of the blocks have the form $\{k, k + 1, k + 4\}$). Find another basic block and so construct a 2-$(13, 3, 1)$ design.

(b) Again let $n = 13$. We aim to construct a 2-$(13, 4, 1)$ design. This must have 13 blocks, so we need one basic block. Find one.

(c) Construct a 2-$(19, 3, 1)$ design via this method.

(d) (a bit harder) Let $n = 11$. Find a basic block which generates a 2-$(11, 5, 2)$ design.

5. Consider the affine plane $\mathbf{A}^2(\mathbf{Z}_7)$. For each pair of points in $\mathbf{A}^2(\mathbf{Z}_7)$, find the equation of the line joining them, and list all the points on that line: (i) $\{(3, 1), (1, 5)\}$, (ii) $\{(4, 1), (4, 6)\}$.

6. Consider the projective plane $\mathbf{P}^2(\mathbf{Z}_7)$. For each pair of points in $\mathbf{P}^2(\mathbf{Z}_7)$, find the equation of the line joining them, and list all the points on that line: (i) $\{[3, 1, 1], [1, 5, 1]\}$, (ii) $\{[4, 1, 1], [4, 6, 1]\}$, (iii) $\{[3, 1, 0], [6, 0, 1]\}$, (iv) $\{[3, 2, 3], [1, 2, 1]\}$ (v) $\{[1, 0, 0], [2, 3, 4]\}$.

7. Let $(X, \mathcal{B})$ be a $2 - (n^2, n, 1)$ design. Let's say blocks B and B' are *parallel* if $B \cap B' = \emptyset$.

   Let $B \in \mathcal{B}$ and $x \in X$ satisfy $x \notin B$. Prove that x lies in a unique block which is parallel to B.

   Deduce that for each block $B \in \mathcal{B}$ there are $(n - 1)$ blocks B' parallel to B.

   Also prove that if B' and B'' are blocks parallel to B then $B' = B''$ or B' and B'' are parallel.

8. Recall that a *symmetric design* is a 2-(v, k, λ) design (where $v > k > 2$) in which the number of blocks b equals v, the number of points.

   Prove that in a symmetric 2-(v, k, λ) design, each point is contained in exactly k blocks. Also, (harder) prove that any two distinct blocks contain exactly λ common points. (If this is too hard, try the special case with $\lambda = 1$.)

9. Let $(X, \mathcal{B})$ be a Steiner triple system of order n, that is a 2-$(n, 3, 1)$ design. We define a binary operation $*$ on X as follows: $x * x = x$ and if $x \neq y$ then $x * y = z$ where $\{x, y, z\}$ is the block containing x and y.

Prove that (i) $x * x = x$, (ii) $x * y = y * x$ and (iii) $x * (x * y) = y$ for all x, $y \in X$. Conversely prove that if one has a finite set X and an operation $*$ on X satisfying (i), (ii) and (iii) then $*$ arises from a Steiner triple system.

Deduce that if there are Steiner triple systems of orders m and n there is also a Steiner triple system of order mn.

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