Combinatorics exam 2012: outline solutions

1(a) Multinomial coefficient:

$$\frac{9!}{2!2!3!} = 15120.$$

1(b)

$$A(t) = \frac{t^2}{1-3t^2+2t^3} = \frac{1}{3(1-t)^2} + \frac{1}{9(1+2t)} - \frac{4}{9(1-t)},$$
$$a_n = \frac{3n-1+(-2)^n}{9}.$$

1(c)

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

Bookwork — see lecture notes.

There are $\binom{2n}{n}$ choices for the n fixed points. The remaining n points are "deranged" in D_n possible ways. Answer:

$$\binom{2n}{n}D_n = \frac{(2n)!}{n!}\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

1(d)

$$1+6x+10x^2+4x^3$$

1(e) Odd parts: $9\,1$, $7\,3$, $7\,1^3$, 5^2, $5\,3\,1^2$, $5\,1^5$, $3^3\,1$, $3^2\,1^4$, $3\,1^7$, 1^{10}. Disticnt parts: 10, $9\,1$, $8\,2$, $7\,3$, $7\,2\,1$, $6\,4$, $6\,3\,1$, $5\,4\,1$, $5\,3\,2$, $4\,3\,2\,1$.

2(a) Bookwork — see lecture notes.

2(b) We count the permutations of $[2m]$ having two cycles of length m. These split $2m$ into two sets of size m; there are $\binom{2m}{m}$ sets of size m and each appears once with its complement. There are $\frac{1}{2}\binom{2m}{m}$ ways to pick these two sets. Each set can be made into a cycle in $(m-1)!$ ways. So there are

$$\frac{1}{2}\binom{2m}{m}(m-1)!^2$$

permutations of $[2m]$ with two cycles of length m, and so this number is $\leq$ the total number of two-cycle permutations which is $s(2m,m)$.

2(c) The usual manipulation gives

$$S(X) = 1 + XS(X) + XS(X)^2.$$

Solving this quadratic gives

$$S(X) = \frac{1 - X - \sqrt{1 - 6X + X^2}}{2X}$$

(a "positive" square root would lead to an absurdity).

3(a) Bookwork (part of proof of Fisher's inequality) — see lecture notes. In fact $M^t M = (k-1)I + J$.

3(b) Set $B_0 = \{P_1, P_2, P_3\}$. Through each P_i there are four blocks — three apart from B_0 itself, but none of those three contains any other P_j. There are thus nine blocks meeting B_0 in one point. There are 12 blocks overall, so that there are $12 - 1 - 9 = 2$ disjoint from B_0.

3(c) Set $B_1 = P_1, P_2, P_3, P_4$. Adapting the proof that each point is an element of $\lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$ blocks, each pair of points is in $\lambda\binom{v-2}{t-2}/\binom{k-2}{t-2}$ blocks (if $t \geq 2$), and so here each pair is in four blocks. Each pair of points in B_1 is in three other blocks, so that $3\binom{4}{2} = 18$ blocks meet B_1 in two points. Each point P_i is in twelve blocks: B_1, three further blocks also through each P_j $(j \neq i)$ (nine in all) and so $12 - 9 - 1 = 2$ blocks meeting B_1 just in P_j. So 8 blocks meet B_1 is just one point. Overall there are 30 blocks, and so $30 - 18 - 8 - 1 = 3$ blocks are disjoint from B_1.

4(a) One gets the recurrence

$$A_n = 2A_{n-1} + A_{n-2}$$

for $n \geq 2$ leading to

$$A(t) = \frac{1}{1 - 2t - t^2} = \frac{\alpha}{(\alpha - \beta)(1 - \alpha t)} + \frac{\beta}{(\beta - \alpha)(1 - \beta)t}$$

where $(\alpha, \beta) = (1 + \sqrt{2}, 1 - \sqrt{2})$. Therefore

$$A_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}}.$$

4(b) (i) just take $x = z^{7/2}$, $y = -z^{3/2}$.

(ii) take $y = -z^{1/2}t$ and $x = z^{1/2}$. This gives

$$\prod_{n=1}^{\infty}(1 - tz^n)(1 - t^{-1}z^{n-1})(1 - z^n) = \sum_{m=-\infty}^{\infty}(-1)^m t^m z^{m(m+1)/2}$$

and so

$$\prod_{n=1}^{\infty}(1 - tz^n)(1 - t^{-1}z^n)(1 - z^n) = \frac{1}{1 - t^{-1}} \sum_{m=-\infty}^{\infty}(-1)^m t^m z^{m(m+1)/2}.$$

Now $(-1)^{-1-m}t^{-1-m}z^{(-1-m)((-1-m)+1)/2} = -(-1)^m t^{-1-m} z^{m(m+1)/2}$ so that

$$\begin{aligned}\sum_{-\infty}^{-1}(-1)^m t^m z^{m(m+1)/2} &= \sum_{0}^{\infty}(-1)^{-1-m}t^{-1-m}z^{(-1-m)((-1-m)+1)/2} \\ &= -\sum_{0}^{\infty}(-1)^m t^{-1-m} z^{m(m+1)/2}\end{aligned}$$

Therefore

$$\begin{aligned}\prod_{n=1}^{\infty}(1 - tz^n)(1 - t^{-1}z^n)(1 - z^n) &= \sum_{m=0}^{\infty}(-1)^m z^{m(m+1)/2}\frac{t^m - t^{-1-m}}{1 - t^{-1}} \\ &= \sum_{m=0}^{\infty}(-1)^m z^{m(m+1)/2}\sum_{k=-m}^{m} t^k.\end{aligned}$$