MAS3042

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING,
COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

June 2003

COMBINATORICS

Module Leader: 9:30 a.m. – 12:30 p.m.

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

Robin Chapman

SECTION A

1. (a) How many anagrams has the word ASSESSMENT? (4)

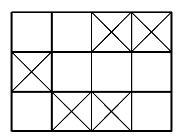
(b) Let the sequence (a_n) be defined by the recurrence: $a_0 = 1$, $a_1 = 2$, $a_{n+2} = 4a_{n+1} - a_n$. Find an explicit expression for the generating function

$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

of the sequence (a_n) and hence find a formula for a_n. (10)

(c) How many numbers between 1 and 1000 inclusive are not divisible by any of the numbers 7, 11 and 13. (6)

(d) Calculate the rook polynomial of (the blank squares of) the following board.



(10)

(e) Define the Ramsey number $R(r, k, l)$ and prove that $R(2, 2, 3) \leq 6$. (10)

[**40**]

SECTION B

2. (a) For integers $n \geq k \geq 1$ the *Stirling number* $S(n,k)$ is defined as the number of partitions of the set $\{1, 2, \ldots, n\}$ into k disjoint nonempty subsets. Prove that $S(n,1) = S(n,n)$ for each n, and that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

for $n > k > 1$. (5)

(b) Prove that

$$\sum_{n=k}^{\infty} S(n,k)t^n = \prod_{r=1}^{k} \frac{t}{1-rt}$$

for each $k \geq 1$. (6)

(c) Define

$$F_r(t) = \sum_{n=1}^{\infty} S(n+r,n)t^n$$

for each integer $r \geq 0$. Prove that

$$F_r(t) = \frac{t}{1-t} F'_{r-1}(t)$$

for $r \geq 1$. Hence, or otherwise, calculate $F_1(t)$. (9)

[**20**]

3. (a) The *Catalan numbers* C_n are defined as follows. A *Catalan path* consists of steps of the forms $(x,y) \mapsto (x+1,y)$ or $(x,y) \mapsto (x,y+1)$ and lies above or on the line $y=x$. Then C_n is the number of Catalan paths from $(0,0)$ to (n,n). Prove that

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

for $n>0$. Deduce that

$$\sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}.$$

(10)

(b) The *Schröder numbers* S_n are defined as follows. A *Schröder path* consists of steps of the forms $(x,y) \mapsto (x+1,y)$, $(x,y) \mapsto (x,y+1)$ or $(x,y) \mapsto (x+1,y+1)$ and lies above or on the line $y=x$. Then S_n is the number of Schröder paths from $(0,0)$ to (n,n). Prove that

$$S_n = S_{n-1} + \sum_{k=1}^{n} S_{k-1}S_{n-k}$$

for $n>0$. Find an explicit expression for

$$\sum_{n=0}^{\infty} S_n t^n.$$

(10)

[**20**]

4. (a) Define a t-(v,k,λ) *design*. Prove a formula for the number of blocks in a t-(v,k,λ) design. (6)

(b) In each case determine whether a design with the given parameters occurs, either by constructing one, or by proving impossibility.

(i) 2-$(6,3,2)$;
(ii) 2-$(17,3,1)$;
(iii) 2-$(25,5,1)$.

(6)

(c) Prove *Fisher's inequality*. This states that, in a 2-(v,k,λ) design with $v>k$, the number of blocks b satisfies $b \geq v$. (8)

[**20**]

5. (a) Prove that the number of partitions of n into exactly k parts equals the number of partitions of n having largest part k. (4)

(b) Let d_n and o_n denote, respectively, the number of partitions of n into distinct parts and into odd parts. Show that

$$\sum_{n=0}^{\infty} d_n t^n = \prod_{k=1}^{\infty} (1+t^k)$$

and

$$\sum_{n=0}^{\infty} o_n t^n = \prod_{k=1}^{\infty} \frac{1}{1-t^{2k-1}}.$$

Hence, or otherwise, prove that $d_n = o_n$ for all n. (9)

(c) The Jacobi triple product formula states that

$$\prod_{n=1}^{\infty} (1+yx^{2n-1})(1+y^{-1}x^{2n-1})(1-x^{2n}) = \sum_{m=-\infty}^{\infty} y^m x^{m^2}.$$

Assuming the Jacobi triple product formula, prove *Euler's pentagonal number theorem*, namely that

$$\prod_{k=1}^{\infty} (1-t^k) = \sum_{m=-\infty}^{\infty} (-1)^m t^{m(3m+1)/2}.$$

(7)

[**20**]