MAS3042

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

June 2005

COMBINATORICS

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

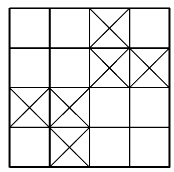
1. (a) How many anagrams has the word HOMOLOGICAL? (6)

   (b) Let the sequence (a_n) be defined by the recurrence: $a_0 = 1$, $a_1 = 0$, $a_{n+2} = 2a_{n+1} - 2a_n$. Find an explicit expression for the generating function

   $$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

   of the sequence (a_n) and hence find a formula for a_n. (14)

   (c) Calculate the rook polynomial of (the blank squares of) the following board.



   (12)

   (d) Find the conjugate of the partition $8\ 6^2\ 5\ 4\ 1$. (6)

   (e) Prove that if each 2-subset of a 10-element set X is coloured red or blue then either there is a 3-subset of X with all its 2-subsets coloured red or a 4-subset of X with all its 2-subsets coloured blue.

   (You may assume that if each 2-subset of a 6-element set Y is coloured red or blue then there is a 3-subset of Y with all its 2-subsets having the same colour.) (12)

[**50**]

SECTION B

2. (a) The *Catalan numbers* C_n are defined as follows. A *Dyck path* consists of steps of the forms $(x,y) \mapsto (x+1,y+1)$ or $(x,y) \mapsto (x+1,y-1)$ and lies above or on the x-axis. Then C_n is the number of Dyck paths from $(0,0)$ to $(2n,0)$. Prove that

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

for $n > 0$. Deduce that

$$\sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}.$$

(10)

(b) The *Motzkin numbers* M_n are defined as follows. A *Motzkin path* consists of steps of the forms $(x,y) \mapsto (x+1,y+1)$, $(x,y) \mapsto (x+1,y)$ or $(x,y) \mapsto (x+1,y-1)$ and lies above or on the x-axis. Then M_n is the number of Motzkin paths from $(0,0)$ to $(n,0)$. Prove that

$$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2}M_{n-k}$$

for $n \geq 2$. Hence calculate M_6. Also find an explicit expression for the generating function

$$\sum_{n=0}^{\infty} M_n t^n.$$

(15)

[25]

3. (a) What is a t-(v,k,λ) *design*? Prove that a t-(v,k,λ) design has

$$\lambda \frac{\binom{v}{t}}{\binom{k}{t}}$$

blocks. Also prove a formula for the number of blocks through each point in a t-(v,k,λ) design.

Deduce that in a 2-$(n,3,1)$ design, either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. (14)

(b) Let B_0 be a block in a 2-$(n,k,1)$ design $(X,\mathcal{B})$. Prove that there are

$$\frac{n(n-1)}{k(k-1)} - \frac{k(n-k)}{k-1} - 1$$

blocks $B \in \mathcal{B}$ with $B \cap B_0 = \emptyset$. (6)

(Hint: for each $a \in B_0$ count the number of blocks B' with $B' \cap B_0 = \{a\}$.)

(c) Find all points on the line through $(3,6)$ and $(5,3)$ in the affine plane $\mathbf{A}^2(Z_7)$. (5)

[25]

4. (a) Prove that the number of self-conjugate partitions of a number n (that is partitions which are their own conjugates) is equal to the number of partitions of n into distinct odd parts.

Hence find an infinite product expansion of the generating function

$$\sum_{n=0}^{\infty} S_n t^n$$

where S_n denotes the number of self-conjugate partitions of n. (12)

(b) Let d_n and o_n denote, respectively, the number of partitions of n into distinct parts and into odd parts. Prove that $d_n = o_n$ for all n. (6)

(c) The Jacobi triple product formula states that

$$\prod_{n=1}^{\infty}(1+yx^{2n-1})(1+y^{-1}x^{2n-1})(1-x^{2n}) = \sum_{m=-\infty}^{\infty} y^m x^{m^2}.$$

Assuming the Jacobi triple product formula, prove that

$$\prod_{k=1}^{\infty}(1-t^k) = \sum_{m=-\infty}^{\infty}(-1)^m t^{m(3m+1)/2}.$$

(7)

[25]