MAS3042

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTER SCIENCE AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2007

COMBINATORICS

Module Leader: Dr M. J. Craven

Duration: 2 HOURS.

The mark for this module is calculated from 75% of the percentage mark for this paper plus 25% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) Consider the sequence (a_n) defined by the recurrence

$$a_0 = 1, a_1 = 0, a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2.$$

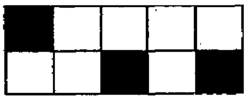
Find an explicit expression for the generating function

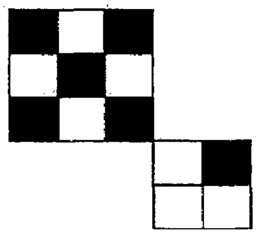
$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

of the sequence (a_n) and hence find a formula for a_n. (12)

(b) How many anagrams has the word DODECAHEDRON? (6)

(c) Give the definition of the rook polynomial of a board B. Calculate the rook polynomials of (the white squares of) each of the following boards:

(i) 

(ii) 

(12)

(d) What is a $t - (v, k, \lambda)$ design? Prove that a $t - (v, k, \lambda)$ design has

$$b = \lambda \frac{\binom{v}{t}}{\binom{k}{t}} \text{ blocks.}$$

(11)

(e) (i) List the partitions of $n = 15$ into distinct parts.

(ii) Find the conjugate of the partition $10\ 7^2\ 4^2\ 3\ 1$. (9)

[50]

SECTION B

2. (a) Let $D(k,n)$ denote the number of permutations π of $[n] = \{1,2,\ldots,n\}$ where $\pi(j) = j$ for exactly k distinct values of j. Calculate the number $D(3,5)$. Using the formula (without proof)

$$D(0,n) = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}$$

find a similar formula for $D(k,n)$. (6)

(b) For integers $n \geq k \geq 1$ the *Stirling number* (of the second kind) $S(n,k)$ is defined to be the number of partitions of $[n]$ into k disjoint non-empty subsets. Prove that $S(n,1) = S(n,n)$ for all n and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $n > k > 1$.

The *Stirling triangle* is a table where the entry on row n and column k is the Stirling number $S(n,k)$. Compute the Stirling triangle up to the fifth row ($n = 5$). (10)

(c) Define the nth *Bell number* to be the sum of the nth row of the Stirling triangle:

$$B_n = \sum_{k=1}^{n} S(n,k)$$

(You may assume $B_0 = 1$.)

Calculate the Bell numbers $B_1, \ldots, B_5$. From Dobinski's formula

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

deduce the exponential generating function

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{e^t - 1}.$$

(9)

[25]

3. (a) Prove that the number of self-conjugate partitions of a number n equals the number of partitions of n into *distinct odd parts.* (7)

(b) Let d_n be the number of partitions of n into *distinct parts.* Show that

$$\sum_{n=0}^{\infty} d_n t^n = \prod_{m=1}^{\infty} (1 + t^m).$$

Let a_n be the number of partitions of n into *odd parts.* Write down an infinite product formula for the generating function, $\sum_{n=0}^{\infty} a_n t^n$.

Hence, or otherwise, prove that $d_n = a_n$ for all n. (10)

(c) The *Jacobi triple product formula* states that

$$\sum_{m=-\infty}^{\infty} y^m x^{m^2} = \prod_{n=1}^{\infty} (1 + yx^{2n-1})(1 + y^{-1}x^{2n-1})(1 - x^{2n}).$$

Assuming this formula, find a formula which expresses

$$\sum_{m=-\infty}^{\infty} t^{m(m+1)/2}.$$

as an infinite product. (8)

[25]

4. (a) Prove a formula for the number of blocks in a $t - (v, k, \lambda)$ design that contain a given point.

From the above formula and from part (d) of question 1, show that for a $2 - (n, 3, 1)$ design either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. (11)

(b) In each of the following cases determine whether a design with the given parameters occurs, either by constructing one, or by proving impossibility:

(i) $2 - (7, 3, 1)$;

(ii) $2 - (11, 3, 1)$;

(iii) $3 - (15, 12, 1)$. (7)

(c) Find the line through $(2, 5)$ and $(9, 2)$ in the Affine plane $A^2(\mathbb{Z}_{11})$, and list all points on that line. (7)

[25]