ECM3721

UNIVERSITY OF EXETER

SCHOOL OF ENGINEERING, COMPUTING AND MATHEMATICS

MATHEMATICAL SCIENCES

May/June 2009

COMBINATORICS

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) How many anagrams has the word HOMOMORPHISMS? (5)

(b) A sequence (a_n) is recursively defined by

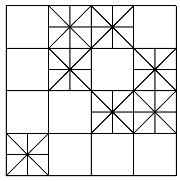
$$a_0 = 1, \quad a_1 = 2, \quad a_2 = -1, \quad a_n = 3a_{n-2} + 2a_{n-3} \ (n \geq 3).$$

Find an explicit expression for the generating function

$$A(X) = \sum_{n=0}^{\infty} a_n X^n$$

and hence, or otherwise, find a general formula for a_n. (15)

(c) Find the rook polynomial of the (blank squares of the) following board:



(15)

(d) How many numbers between 1001 and 2000 inclusive are divisible by none of the three numbers 7, 11 and 13? (10)

(e) Find all partitions of 16 which are self-conjugate. (A partition is self-conjugate if it is equal to its own conjugate.) (5)

[**50**]

SECTION B

2. (a) The Stirling number $S(n, k)$ is defined as the number of partitions of an n-element set into k non-empty subsets. Prove that $S(n, n) = S(n, 1) = 1$ and that

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

whenever $1 < k < n$. Also prove that

$$\sum_{n=k}^{\infty} S(n, k)X^n = \prod_{j=1}^{k} \frac{X}{1 - jX}$$

for all positive integers k. (12)

(b) The *Motzkin numbers* M_n are defined as follows. A *Motzkin path* consists of steps of the forms $(x, y) \mapsto (x + 1, y + 1)$, $(x, y) \mapsto (x + 1, y)$ or $(x, y) \mapsto (x + 1, y - 1)$ and lies above or on the x-axis. Then M_n is the number of Motzkin paths from $(0, 0)$ to $(n, 0)$ (as a special case $M_0 = 1$). Prove that

$$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2}M_{n-k}$$

for $n \geq 2$. Hence, or otherwise, calculate M_7. Also find an explicit expression for the generating function

$$M(X) = \sum_{n=0}^{\infty} M_n X^n.$$

(13)

[**25**]

3. (a) Let d_n and o_n denote, respectively, the number of partitions of n into distinct parts and into odd parts. Prove that

$$\sum_{n=0}^{\infty} d_n t^n = \prod_{k=1}^{\infty}(1+t^k)$$

and

$$\sum_{n=0}^{\infty} o_n t^n = \prod_{k=1}^{\infty} \frac{1}{1-t^{2k-1}}.$$

Hence, or otherwise, prove that $d_n = o_n$ for all n. (10)

(b) The Jacobi triple product formula states that

$$\prod_{n=1}^{\infty}(1+yx^{2n-1})(1+y^{-1}x^{2n-1})(1-x^{2n}) = \sum_{m=-\infty}^{\infty} y^m x^{m^2}.$$

Assuming the Jacobi triple product formula express

$$\sum_{m=0}^{\infty} t^{m(m+1)/2}$$

as an infinite product. (9)

(c) Let $P(t) = \sum_{n=0}^{\infty} p_n t^n = \prod_{m=1}^{\infty}(1-t^m)^{-1}$. Prove that

$$\frac{P'(t)}{P(t)} = \frac{d}{dt}\log P(t) = \sum_{n=1}^{\infty} \sigma(n) t^{n-1}$$

where $\sigma(n)$ is the sum of the positive integer divisors of n. (6)

[25]

4. (a) Prove that a t-(v,k,λ) design has

$$\lambda \frac{\binom{v}{t}}{\binom{k}{t}}$$

blocks and find a formula for the number of blocks containing a given point in such a design. (10)

(b) A Steiner triple system is a 2-$(n,3,1)$ design. Prove that if a Steiner triple system exists with n points then $n \equiv 1$ or $3 \pmod 6$. (6)

(c) Let B_0 be a block in a Steiner triple system on n points. How many blocks B in the system have the property that $B \cap B_0 = \emptyset$? (9)

[25]