ECM3721

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May/June 2011

Combinatorics

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) How many anagrams has the word PSEUDOSPHERE? (5)

(b) Let the sequence (a_n) be defined by the recurrence: $a_0 = 1$, $a_1 = 2$, and $a_{n+2} = 4a_{n+1} - 13a_n$ for $n \geq 0$. Find an explicit expression for the generating function

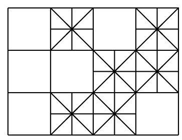
$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

of the sequence (a_n) and hence find a formula for a_n. (15)

(c) A *permutation* of the set $[n] = \{1, 2, \ldots, n\}$ is a list $(b_1, b_2, \ldots, b_n)$ of the elements of $[n]$ with each element occurring once. It is a *derangement* if $b_j \neq j$ for all $j \in [n]$. Use the inclusion-exclusion principle to determine how many derangements of [6] there are.

How many permutations of [6] have the property that $b_j = j$ for exactly three different values of j? (15)

(d) Find the rook polynomial of (the blank squares) of the following board:



(10)

(e) Find the conjugate of the partition $9\ 6^2\ 4\ 3\ 2$. (5)

[**50**]

SECTION B

2. (a) The *Stirling numbers of the first kind* $s(n,k)$ are defined for integers $n \geq k \geq 0$ and satisfy

   - $s(n,n) = 1$ for all $n \geq 0$,
   - $s(n,0) = 0$ for all $n \geq 1$ and
   - $s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k)$ if $0 < k < n$

   Compute $s(5,k)$ for all k with $0 \leq k \leq 5$.

   Prove that

   $$s(n,2) = (n-1)! \sum_{j=1}^{n-1} \frac{1}{j}$$

   for $n \geq 2$. (10)

   (b) The *Schröder numbers* S_n are defined as follows. A *Schröder path* starts at $(0,0)$ with allowable steps

   - $(x,y) \mapsto (x+1, y+1)$,
   - $(x,y) \mapsto (x+1, y-1)$,
   - $(x,y) \mapsto (x+2, y)$

   and lies completely on or above the x-axis. Then S_n is the number of Schröder paths finishing at $(2n,0)$ (also $S_0 = 1$). Prove that

   $$S_n = S_{n-1} + \sum_{k=1}^{n} S_{k-1} S_{n-k}$$

   for $n > 0$.

   Find an explicit expression for the generating function

   $$\sum_{n=0}^{\infty} S_n X^n.$$

   (15)

   [**25**]

3. (a) Prove that a t-$(v,k,1)$ design has

$$\frac{\binom{v}{t}}{\binom{k}{t}}$$

blocks and determine a formula for the number of blocks which contain a given point.

Prove that if a 2-$(v,4,1)$ design exists then $v \equiv 1$ or $4 \pmod{12}$. (17)

(b) Let B_0 be a fixed block in a 2-$(v,4,1)$ design. Prove that there are

$$\frac{v(v-1)}{12} - \frac{4(v-4)}{3} - 1$$

blocks B with $B \cap B_0 = \emptyset$. (8)

[**25**]

4. (a) Prove that the number of self-conjugate partitions of a number n (that is partitions which are their own conjugates) is equal to the number of partitions of n into distinct odd parts. (8)

(b) The *rank* of a partition is the largest part minus the number of parts. For instance, the partition $5\ 3^2\ 2^2\ 1$ has rank $5-6=-1$. Prove that the number of partitions of n with rank k equals the number of partitions of n with rank $-k$. (7)

(c) Let $P(t) = \sum_{n=0}^{\infty} p_n t^n = \prod_{m=1}^{\infty}(1-t^m)^{-1}$ be the generating function for the partition numbers. Prove that

$$\frac{P'(t)}{P(t)} = \frac{d}{dt}\log P(t) = \sum_{n=1}^{\infty} \sigma(n) t^{n-1}$$

where $\sigma(n)$ is the sum of the positive integer divisors of n. (10)

[**25**]