ECM3721

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May/June 2012

Combinatorics

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) How many anagrams has the word DISMISSED? (5)

(b) Let the sequence (a_n) be defined by the recurrence: $a_0 = 0$, $a_1 = 0$, $a_2 = 1$ and $a_n = 3a_{n-2} - 2a_{n-3}$ for $n \geq 3$. Find an explicit expression for the generating function

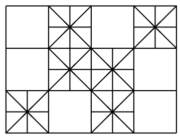
$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

of the sequence (a_n) and hence find a formula for a_n. (15)

(c) A *permutation* of the set $[n] = \{1, 2, \ldots, n\}$ is a bijection $\pi : [n] \to [n]$. It is a *derangement* if $\pi(j) \neq j$ for all $j \in [n]$. Use the inclusion-exclusion principle to prove a formula for the number D_n of derangements of $[n]$.

How many permutations π of $[2n]$ have the property that $\pi(j) = j$ for exactly n values of j? (15)

(d) Find the rook polynomial of (the blank squares of) the following board:



(9)

(e) Find all partitions of 10 (i) into odd parts, (ii) into distinct parts.

(6)

[**50**]

SECTION B

2. (a) The *Stirling number of the first kind* $s(n,k)$ is defined as the number of permutations of $[n] = \{1, 2, \dots, n\}$ having k cycles. Prove that $s(n, 1) = (n-1)!$, $s(n, n) = 1$ and

$$s(n,k) = s(n-1, k-1) + (n-1)s(n-1, k)$$

whenever $1 < k < n$. (11)

(b) Prove that

$$s(2m, 2) \geq \frac{1}{2}\binom{2m}{m}(m-1)!^2$$

for all positive integers m. (6)

(c) The Schröder numbers S_n satisfy $S_0 = 1$ and

$$S_n = 2S_{n-1} + \sum_{k=1}^{n-1} S_k S_{n-k-1}$$

for $n \geq 1$. Find a quadratic equation satisfied by the generating function

$$S(X) = \sum_{n=0}^{\infty} S_n X^n$$

and solve it to obtain an explicit formula for $S(X)$. (8)

[25]

3. Recall that a t-(v, k, λ) design has $\lambda\binom{v}{t}/\binom{k}{t}$ blocks and that any given point is an element of $\lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$ blocks.

(a) Let $(X, \mathcal{B})$ be a 2-$(v, k, 1)$ design. Let its set of points be $X = \{P_1 \dots, P_v\}$ and its set of blocks be $\mathcal{B} = \{B_1 \dots, B_b\}$. Then its incidence matrix is the b-by-v matrix M whose (i, j)-entry is

$$m_{i,j} = \begin{cases} 1 & \text{if } P_j \in B_i, \\ 0 & \text{if } P_j \notin B_i. \end{cases}$$

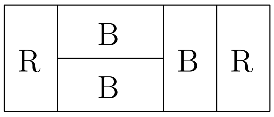
Prove that $M^t M = \alpha I + J$ where I is the v-by-v identity matrix, J is the v-by-v all-one matrix and α is some number. (7)

(b) Let B_0 be a fixed block in a 2-$(9, 3, 1)$ design. Prove that there are two blocks B with $B \cap B_0 = \emptyset$. (6)

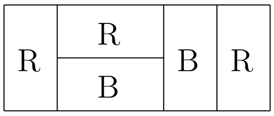
(c) Let B_1 be a fixed block in a 3-$(10, 4, 1)$ design. Prove that there are 18 blocks B with $|B \cap B_1| = 2$. How many blocks B have $|B \cap B_1| = 1$? How many blocks B have $B \cap B_1 = \emptyset$? (12)

[25]

4. (a) You are given a supply of 2×1 dominoes, and are asked to tile a $2 \times n$ rectangle. The dominoes come in two colours: red and blue, but while dominoes placed in a vertical position are allowed to be red or blue, those placed in a horizontal position must be blue. For instance,



is allowed, but



is not. Determine a formula for A_n, the number of allowable domino tilings for the $2 \times n$ rectangle. (You may take $A_0 = 1$.) (12)

(b) The Jacobi triple product states that

$$\prod_{n=1}^{\infty}(1 + yx^{2n-1})(1 + y^{-1}x^{2n-1})(1 - x^{2n}) = \sum_{m=-\infty}^{\infty} y^m x^{m^2}.$$

Use the Jacobi triple product to prove that

(i)

$$\prod_{n=1}^{\infty}(1 - z^{7n-5})(1 - z^{7n-2})(1 - z^{7n}) = \sum_{m=-\infty}^{\infty}(-1)^m z^{m(7m+3)/2},$$

(3)

(ii)

$$\prod_{n=1}^{\infty}(1 - z^n)(1 - z^n t)(1 - z^n t^{-1}) = \sum_{m=0}^{\infty}(-1)^m z^{m(m+1)/2} \sum_{r=-m}^{m} t^r.$$

(10)

[25]