ECM3721

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May 2013

Combinatorics

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) How many anagrams has the word SYLLOGISMS? (5)

(b) A *tromino* is a 3-by-1 rectangle. Let A_n denote the number of ways of covering a 3-by-n rectangle with n trominoes (which may be placed vertically or horizontally). Taking $A_0 = 1$, find a linear recurrence for A_n of the form

$$A_n = b_1 A_{n-1} + b_2 A_{n-2} + b_3 A_{n-3} \qquad (n \geq 3)$$

and use it to find an explicit formula for the generating function

$$A(t) = \sum_{n=0}^{\infty} A_n t^n.$$

(12)

(c) The *Stirling numbers of the second kind* $S(n, k)$ satisfy the initial conditions

$$S(n, 1) = S(n, n) = 1 \qquad (n \geq 1)$$

and the recurrence

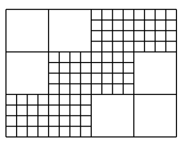
$$S(n, k) = kS(n-1, k) + S(n-1, k-1) \qquad (n > k > 1).$$

Compute all the $S(n, k)$ for $1 \leq k \leq n \leq 5$. Prove that

$$\sum_{n=3}^{\infty} S(n, 3) t^n = \frac{t^3}{(1-t)(1-2t)(1-3t)}.$$

(14)

(d) Find the rook polynomial of (the blank squares) of the following board:



(10)

(e) Find all self-conjugate partitions of 19. (9)

[50]

SECTION B

2. (a) The Catalan numbers satisfy the recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \qquad (n \geq 0)$$

with the initial condition $C_0 = 1$. Prove that $C_{2n+1} - C_n$ is always an even number. Find five different n such that C_n is odd, and prove that there are infinitely many n with C_n odd. (11)

(b) Call a path from $(0,0)$ to $(n,0)$ *admissible* if it lies entirely on or above the x-axis, and each of its steps are of the following forms

- $(x,y) \to (x+1, y+1)$,
- $(x,y) \to (x+1, y-1)$,
- $(x,y) \to (x+3, y)$.

Let U_n denote the number of admissible paths from $(0,0)$ to $(n,0)$ taking $U_0 = 1$. Show that $U_1 = 0$ and $U_2 = 1$, and find a recurrence for U_n valid for $n \geq 3$. Use this recurrence to find an explicit formula for the generating function

$$U(t) = \sum_{n=0}^{\infty} U_n t^n.$$

(14)

[25]

3. (a) Let $(X, \mathcal{B})$ be a t-$(v,k,1)$ design. Prove that it has $\binom{v}{t}/\binom{k}{t}$ blocks and that any given point is an element of $\binom{v-1}{t-1}/\binom{k-1}{t-1}$ blocks. (8)

(b) Let B_0 be a fixed block in a 2-$(21,5,1)$ design. Prove that every other block B in the design satisfies $|B \cap B_0| = 1$. (7)

(c) Let B_1 be a fixed block in a 3-$(22,6,1)$ design. Prove that there are 60 blocks B with $|B \cap B_1| = 2$. How many blocks B have $|B \cap B_1| = 1$? How many blocks B have $B \cap B_1 = \emptyset$? (10)

[25]

4. (a) Prove that the number of partitions of n into k parts equals the number of partitions of n with largest part k. (5)

(b) Let $\mathcal{A}_k$ denote the set of all partitions which have both (i) exactly k parts and (ii) largest part k. Write down all elements of $\mathcal{A}_3$ and their Ferrers diagrams. State and prove a formula for the number of elements of $\mathcal{A}_k$. (10)

(c) Let $p_k(n)$ denote the number of partitions of n which have exactly k parts, and all of which are distinct. Prove that

$$\sum_{n=0}^{\infty} p_k(n) t^k = \frac{f_k(t)}{\prod_{j=1}^{k}(1 - t^j)}$$

where $f_k(t)$ is a polynomial that you should determine. (10)

[**25**]