ECM3721

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

May 2014

Combinatorics

Module Leader: Robin Chapman

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) How many anagrams has the word LOLLIPOPS? (6)

(b) Let the sequence (a_n) be defined by the recurrence: $a_0 = 1$, $a_1 = 0$, $a_2 = 0$ and $a_n = 3a_{n-1} - 4a_{n-3}$ for $n \geq 3$. Find an explicit expression for the generating function

$$A(t) = \sum_{n=0}^{\infty} a_n t^n$$

of the sequence (a_n) and hence find a formula for a_n. (16)

(c) We consider paths from the point $(0, 0)$ to the point (n, n) where we are permitted

- Right steps: from (a, b) to $(a + 1, b)$,
- Up steps: from (a, b) to $(a, b + 1)$,
- Diagonal steps: from (a, b) to $(a + 1, b + 1)$.

For each integer $0 \leq k \leq n$ determine how many paths from $(0, 0)$ to (n, n) there are containing exactly k Diagonal steps.

Determine the total number of admissible paths from $(0, 0)$ to $(4, 4)$. (10)

(d) Determine how many numbers between 1001 and 2000 inclusive are divisible by neither 5 nor 7 nor 11. (10)

(e) Find all partitions of 12 into (i) distinct parts, (ii) odd parts, (iii) distinct odd parts. (8)

[**50**]

SECTION B

2. (a) The *Stirling numbers of the second kind* satisfy the initial conditions $S(n,1) = S(n,n) = 1$ and the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

for $1 < k < n$. Define polynomials $T_n(x)$ by

$$T_n(x) = \sum_{k=1}^{n} S(n,k)x^k.$$

Prove that

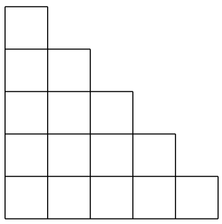
$$T_{n+1}(x) = x(T_n(x) + T_n'(x))$$

for $n \geq 1$. Deduce that if $U_n(x) = e^x T_n(x)$ then

$$U_{n+1}(x) = xU_n'(x).$$

(10)

(b) Let B_n denote the "staircase board" consisting of rows of squares of lengths $1, 2, \ldots, n$ aligned at their left ends. For example B_5 is illustrated below.



Let $A(n,k)$ denote the number of arrangements of k non-attacking rooks on B_n. (By convention we take $A(0,0) = 1$ and $A(n,k) = 0$ whenever $k > n$.) Clearly $A(n,0) = 1$. By separating out the positions with a rook on the final row from those without a rook in the final row prove that for $n > 0$ and $k > 0$

$$A(n,k) = (n-k+1)A(n-1,k-1) + A(n-1,k).$$

Deduce that for $0 \leq k \leq n$,

$$A(n,k) = S(n+1, n+1-k).$$

(15)

3. (a) You have a supply of cards, each coloured red, blue or green. You arrange n of these cards in a row. Such an arrangement is called *admissible* if

- no two blue cards are adjacent, and
- no green card has a red or blue card to its right.

For example $RBRRBGGG$ is an admissible arrangement, but $RBBRBGGG$ and $RBRRBGBG$ are not.

Let r_n denote the number of admissible n-card arrangements having a red card as the right-most card. Similarly let b_n and g_n respectively denote the numbers of admissible n-card arrangements having a blue or green card respectively as the right-most card. Define

$$R(t) = \sum_{n=1}^{\infty} r_n t^n, \quad B(t) = \sum_{n=1}^{\infty} b_n t^n \quad \text{and} \quad G(t) = \sum_{n=1}^{\infty} g_n t^n.$$

Prove that

$$R(t) = t + tR(t) + tB(t)$$

and give similar formulas for $B(t)$ and $G(t)$. Hence find an explicit formula for $G(t)$ and use that to find an explicit formula for g_n. (20)

(b) A certain combinatorial sequence (a_n) satisfies $a_0 = a_1 = 1$ and

$$a_n = a_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-2-k}.$$

Find an explicit formula for the generating function

$$A(t) = \sum_{n=0}^{\infty} a_n t^n.$$

(5)

[25]

4. (a) Let X be the set of points, and $\mathcal{B}$ be the set of points in a t-$(v, k, 1)$ design. Let A be a subset of X with $|A| = r \leq t$. Prove that there are exactly

$$\frac{\binom{v-r}{t-r}}{\binom{k-r}{t-r}}$$

blocks $B \in \mathcal{B}$ with $A \subseteq B$. (6)

(b) Using the inclusion-exclusion principle or otherwise, prove that if B_0 is a block in a $t - (v, k, 1)$ design then there are

$$\sum_{r=0}^{t}(-1)^r \binom{k}{r} \frac{\binom{v-r}{t-r}}{\binom{k-r}{t-r}} + \sum_{r=t+1}^{k} (-1)^r \binom{k}{r}$$

blocks B with $B \cap B_0 = \emptyset$. (10)

(c) A *Steiner quaduple system of order* n is a 3-$(n, 4, 1)$ design. Prove that if there is a Steiner quadruple system of order n then either $n \equiv 2$ or $n \equiv 4$ (mod 6). Prove that in a Steiner quadruple system of order 8, the complement of each block is also a block. (9)

[**25**]