Sums of k-th powers

Robin Chapman

19 November 2015

In the first year you meet formulae like

$$\sum_{m=1}^{n} m = \frac{n^2}{2} + \frac{n}{2},$$

$$\sum_{m=1}^{n} m^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

and

$$\sum_{m=1}^{n} m^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^4}{4}.$$

The aim of presenting these appears to be an attempt to put you off proof by induction for life. I give here a systemtic way of finding such formulae. This is fairly tangential to combinatorics proper, but does give a nice illustration of the use of exponential generating functions.

I'll write

$$S_k(n) = 1^k + 2^k + \cdots + n^k = \sum_{m=1}^{n} m^k.$$

For each fixed natural number n I'll consider the exponential generating function defined by

$$F_n(t) = \sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!}.$$

We calculate

$$F_n(t) = \sum_{k=0}^{\infty} \sum_{m=1}^{n} m^k \frac{t^k}{k!} = \sum_{m=1}^{n} \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} = \sum_{m=1}^{n} e^{mt}.$$

This last sum is a geometric progression with n term, initial term e^t and common ratio e^t. By the formula for sums of finite geometric progressions

$$F_n(t) = e^t \frac{(e^t)^n - 1}{e^t - 1} = \frac{e^{nt} - 1}{1 - e^{-t}}.$$

I'll rewrite this, apparently perversely, as

$$F_n(t) = \frac{e^{nt}-1}{t}\frac{t}{1-e^{-t}} = \frac{e^{nt}-1}{t}\left(\frac{1-e^{-t}}{t}\right)^{-1}.$$

The point is that

$$\frac{1-e^{-t}}{t} = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{t^{r-1}}{r!} = \sum_{r=0}^{\infty}(-1)^r\frac{t^r}{(r+1)!}$$

is a power series with constant term 1, and is independent of n. Therefore so is its reciprocal

$$\frac{t}{1-e^{-t}} = \sum_{r=0}^{\infty} c_r t^r.$$

We don't know what the c_r are, but $c_0 = 1$ and comparing coefficients of x^s in the product

$$1 = \left(\sum_{r=0}^{\infty}(-1)^r\frac{t^r}{(r+1)!}\right)\left(\sum_{r=0}^{\infty} c_r t^r\right)$$

gives

$$0 = \sum_{r=0}^{s}(-1)^r\frac{c_{s-r}}{(r+1)!}$$

for $s \geq 1$. Equivalently,

$$c_s = \sum_{r=1}^{s}(-1)^{r-1}\frac{c_{s-r}}{(r+1)!}.$$

This gives

$$c_1 = \frac{c_0}{2} = \frac{1}{2},$$

$$c_2 = \frac{c_1}{2} - \frac{c_0}{6} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12},$$

$$c_3 = \frac{c_2}{2} - \frac{c_1}{6} + \frac{c_0}{24} = \frac{1}{24} - \frac{1}{12} + \frac{1}{24} = 0$$

and

$$c_4 = \frac{c_3}{2} - \frac{c_2}{6} + \frac{c_1}{24} - \frac{c_0}{120} = -\frac{1}{72} + \frac{1}{48} - \frac{1}{120} = -\frac{1}{720}$$

etc.

Also

$$\frac{e^{nt}-1}{t} = \sum_{j=1}^{\infty} n^j\frac{t^{j-1}}{j!} = \sum_{j=0}^{\infty} n^{j+1}\frac{t^j}{(j+1)!}.$$

Then $S_k(n)$ is $k!$ times the t^k-coefficient of

$$\left(\sum_{r=0}^{\infty} c_r t^t\right)\left(\sum_{k=0}^{\infty} n^{j+1}\frac{t^j}{(j+1)!}\right).$$

Therefore

$$S_k(n) = k!\sum_{r=0}^{k} c_r \frac{n^{k-r+1}}{(k-r+1)!}.$$

This formula is usually attributed to Johann Faulhaber (1580–1635). As an example,

$$\begin{aligned} S_4(n) &= 4!\left(c_0\frac{n^5}{5!} + c_1\frac{n^4}{4!} + c_2\frac{n^3}{3!} + c_3\frac{n^2}{2!} + c_4\frac{n}{1!}\right) \\ &= c_0\frac{n^5}{5} + c_1 n^4 + 4c_2 n^2 + 12c_3 n^2 + 24c_4 n \\ &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}. \end{aligned}$$

So, what are the numbers c_r? It's a nice exercise to prove that $c_r = 0$ whenever r is odd, except when $r = 1$. In general the c_r are related to the so-called *Bernoulli numbers* which you might like to look up.