# Infinite products of power series

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In this course I have mainly been treating power series "formally", that is not worrying about for which values of the argument they converge for. This is fine, since one can add, subtract and multiply them without bothering about convergence, and one can even give a power series for $1/f(t)$ where $f(t) = \sum_{n=0}^{\infty} a_n t^n$ as long as $a_0 \neq 0$.

In partition theory I use infinite products of power series. The question arises as to whether one can make sense of an infinite product of power series as a power series, at least in a formal sense. The answer is not always, but in favourable circumstances yes.

I claim that infinite products of the form

$$\prod_{m=1}^{\infty} (1 + q^{k_m} g_m(t))$$

make sense, provided that $(k_m)$ is a sequence of nonnegative integers diverging to infinity. To see this consider the finite product

$$\prod_{m=1}^{M} (1 + q^{k_m} g_m(t)) = \sum_{n=0}^{\infty} a_{M,n} t^n.$$

Then

$$\sum_{n=0}^{\infty} a_{M+1,n} t^n = (1 + q^{k_{M+1}} g_{M+1}(t)) \sum_{n=0}^{\infty} a_{M,n} t^n.$$

What can we say about the $a_{M+1,n}$? Well for small enough $n$, $a_{M+1,n} = a_{M,n}$; this is certainly true for $n < k_{M+1}$. For a given $n \in \mathbf{N}$, as $k_m \to \infty$, then for large enough $M$, $k_{M+1} > n$ and so $a_{M+1,n} = a_{M,n}$. Thus for each $n$ the sequence $(a_{M,n})_{M=1}^{\infty}$ is eventually constant. Call this constant $a_{\infty,n}$. Then we define

$$\prod_{m=1}^{\infty} (1 + q^{k_m} g_m(t)) = \sum_{n=0}^{\infty} a_{\infty,n} t^n.$$

To see this in action, take the classic product

$$\prod_{m=1}^{\infty} \frac{1}{1-t^m} = \prod_{m=1}^{\infty} \sum_{r=0}^{\infty} t^{rm}.$$

This falls into the setup above as $1/(1-t^m) = 1 + t^m g_m(t)$ where $g_m(t) = 1/(1-t^m)$. The first few finite products $\prod_{m=1}^{M} 1/(1-t^m)$ are:

$$\begin{aligned}
&1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + \cdots,\\
&1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 + 6t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + 8t^7 + 10t^8 + 12t^9 + 14t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 + 18t^9 + 23t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 10t^6 + 13t^7 + 18t^8 + 23t^9 + 30t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 14t^7 + 20t^8 + 26t^9 + 35t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 21t^8 + 28t^9 + 38t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 22t^8 + 29t^9 + 40t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 22t^8 + 30t^9 + 41t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 22t^8 + 30t^9 + 42t^{10} + \cdots,\\
&1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 22t^8 + 30t^9 + 42t^{10} + \cdots.
\end{aligned}$$

As we see, the coefficient of each $t^n$ "settles down" to a limiting value — this is the coefficient of $t^n$ in the infinite product.

A similar argument shows that infinite sums

$$\sum_{m=1}^{\infty} t^{k_m} g_m(t)$$

make sense as long as $k_m \to \infty$ and each $g_m(t)$ is a power series.