Steiner triple systems

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In the lectures I gave a construction of a Steiner triple system (STS) of order n whenever $n \equiv 3 \pmod 6$. I stated that there was a construction when $n \equiv 1 \pmod 6$. I outline a construction here, but leave proving that it works as an exercise. This construction is taken from *A Course in Combinatorics* by van Lint and Wilson (CUP, 1992).

We write $n = 6m + 1$ with m a positive integer. The construction works modulo $2m$, that is in the ring $\mathbf{Z}_{2m} = \{0, 1, \dots, 2m - 1\}$ where addition, subtraction and multiplication are defined modulo $2m$. The $6m + 1$ points are denoted a_j, b_j, c_j (for $j \in \mathbf{Z}_{2m}$) and d. The blocks are

- $\{a_k, b_k, c_k\}$, for $k = 0, \dots, m - 1$;
- $\{a_k, b_{m+k}, d\}$, $\{b_k, c_{m+k}, d\}$, $\{c_k, a_{m+k}, d\}$, for $k = 0, \dots, m - 1$;
- $\{a_k, b_{j+k}, b_{-j+k}\}$, $\{b_k, c_{j+k}, c_{-j+k}\}$, $\{c_k, a_{j+k}, a_{-j+k}\}$, for $j = 1, \dots, m-1$ and $k = 0, \dots, m - 1$;
- $\{a_{m+k}, b_{j+k}, b_{1-j+k}\}$, $\{b_{m+k}, c_{j+k}, c_{1-j+k}\}$, $\{c_{m+k}, a_{j+k}, a_{1-j+k}\}$, for $j = 1, \dots, m$ and $k = 0, \dots, m - 1$.

The verification that any two points lie in a unique block is straightforward but rather tedious.