Computing the Jordan Canonical Form

Let A be an n by n square matrix. If its characteristic equation $\chi_A(t) = 0$ has a repeated root then A may not be diagonalizable, so we need the Jordan Canonical Form. Suppose λ is an eigenvalue of A, with multiplicity r as a root of $\chi_A(t) = 0$. The the vector $\boldsymbol{v}$ is an eigenvector with eigenvalue λ if $A\boldsymbol{v} = \lambda\boldsymbol{v}$ or equivalently

$$(A - \lambda I)\boldsymbol{v} = 0.$$

The trouble is that this equation may have fewer then r linearly independent solutions for $\boldsymbol{v}$. So we generalize and say that $\boldsymbol{v}$ is a *generalized eigenvector* with eigenvalue λ if

$$(A - \lambda I)^k\boldsymbol{v} = 0$$

for some positive integer k. Now one can prove that there are exactly r linearly independent generalized eigenvectors. Finding the Jordan form is now a matter of sorting these generalized eigenvectors into an appropriate order.

To find the Jordan form carry out the following procedure for each eigenvalue λ of A. First solve $(A - \lambda I)\boldsymbol{v} = 0$, counting the number r_1 of linearly independent solutions. If $r_1 = r$ good, otherwise $r_1 < r$ and we must now solve $(A - \lambda I)^2\boldsymbol{v} = 0$. There will be r_2 linearly independent solutions where $r_2 > r_1$. If $r_2 = r$ good, otherwise solving $(A - \lambda I)^3\boldsymbol{v} = 0$ gives $r_3 > r_2$ linearly independent solutions, and so on. Eventually one gets $r_1 < r_2 < \cdots < r_{N-1} < r_N = r$. The number N is the size of the largest Jordan block associated to λ, and r_1 is the total number of Jordan blocks associated to λ. If we define $s_1 = r_1$, $s_2 = r_2 - r_1$, $s_3 = r_3 - r_2, \ldots, s_N = r_N - r_{N-1}$ then s_k is the number of Jordan blocks of size at least k by k associated to λ. Finally put $m_1 = s_1 - s_2$, $m_2 = s_2 - s_3, \ldots, m_{N-1} = s_{N-1} - s_N$ and $m_N = s_N$. Then m_k is the number of k by k Jordan blocks associated to λ. Once we've done this for all eigenvalues then we've got the Jordan form!

To find P such that $J = P^{-1}AP$ is the Jordan form then we need to work a bit harder. We do the following for each eigenvalue λ. First find the Jordan block sizes associated to λ by the above process. Put them in decreasing order $N_1 \geq N_2 \geq N_3 \geq \cdots \geq N_k$. Now find a vector $\boldsymbol{v}_{1,1}$ such that $(A-\lambda I)^{N_1}\boldsymbol{v}_{1,1} = 0$ but $(A-\lambda I)^{N_1-1}\boldsymbol{v}_{1,1} \neq 0$. Define $\boldsymbol{v}_{1,2} = (A-\lambda I)\boldsymbol{v}_{1,1}$, $\boldsymbol{v}_{1,3} = (A - \lambda I)\boldsymbol{v}_{1,2}$, and so on until we get $\boldsymbol{v}_{1,N_1}$. We can't go further as $(A-\lambda I)\boldsymbol{v}_{1,N_1} = 0$. If we only have one block we're OK, otherwise we can find a vector $\boldsymbol{v}_{2,1}$ such that $(A - \lambda I)^{N_2}\boldsymbol{v}_{2,1} = 0$, $(A - \lambda I)^{N_2-1}\boldsymbol{v}_{2,1} \neq 0$ and **(this**

is important!) $\boldsymbol{v}_{2,1}$ is not linearly dependent on $\boldsymbol{v}_{1,1}, \ldots, \boldsymbol{v}_{1,N_1}$. Define $\boldsymbol{v}_{2,2} = (A - \lambda I)\boldsymbol{v}_{2,1}$ etc., until we get to $\boldsymbol{v}_{2,N_2}$. If $k = 2$ this is the end, if not then choose $\boldsymbol{v}_{3,1}$ with $(A - \lambda I)^{N_3}\boldsymbol{v}_{3,1} = 0$, $(A - \lambda I)^{N_3-1}\boldsymbol{v}_{3,1} \neq 0$ and $\boldsymbol{v}_{3,1}$ not linearly dependent on $\boldsymbol{v}_{1,1}, \ldots, \boldsymbol{v}_{1,N_1}, \boldsymbol{v}_{2,1}, \ldots \boldsymbol{v}_{2,N_2}$. Keep going! Eventually we get r linearly independent vectors $\boldsymbol{v}_{1,1}, \boldsymbol{v}_{1,2}, \ldots, \boldsymbol{v}_{k,N_k}$. Let

$$P_\lambda = (\boldsymbol{v}_{k,N_k} \;\cdots\; \boldsymbol{v}_{1,1})$$

be the n by r matrix whose columns are these vectors in **reverse** order. Once we've done this for all eigenvalues λ stick the matrices P_λ together horizontally to get an n by n matrix P. Then P will be non-singular, and $P^{-1}AP = J$, the Jordan form.

A worked example

To illustrate this method, I give a reasonably sized example (6 by 6) which I hope will make things clear, and I hope is safely too big come up on any exam! I have used MAPLE in the computations; only a truly hardy soul would try this one by hand!

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -8 & 4 & -3 & 1 & -3 \\ -3 & 13 & -8 & 6 & 2 & 9 \\ -2 & 14 & -7 & 4 & 2 & 10 \\ 1 & -18 & 11 & -11 & 2 & -6 \\ -1 & 19 & -11 & 10 & -2 & 7 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\chi_A(t) = t^6 + 3t^5 - 10t^3 - 15t^2 - 9t - 2 = (t+1)^5(t-2)$$

and so its eigenvalues are -1 with multiplicity 5, and 2 with multiplicity 1. I'll deal with $\lambda = -1$ first. We first solve $(A + I)\boldsymbol{v} = 0$. The matrix

$$A + I = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -7 & 4 & -3 & 1 & -3 \\ -3 & 13 & -7 & 6 & 2 & 9 \\ -2 & 14 & -7 & 5 & 2 & 10 \\ 1 & -18 & 11 & -11 & 3 & -6 \\ -1 & 19 & -11 & 10 & -2 & 8 \end{pmatrix}$$

has REF

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 2 & 3/2 \\ 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $(A+I)\boldsymbol{v}$ has 2 linearly independent solutions, i.e., $r_1 = 2$. As $r_1 < r = 5$ then we must solve $(A+I)^2\boldsymbol{v} = 0$. Now

$$(A+I)^2 = \begin{pmatrix} 1 & -1 & 0 & 1 & -2 & -3 \\ -2 & -16 & 9 & -11 & 4 & -3 \\ -1 & 37 & -18 & 17 & 2 & 21 \\ 1 & 35 & -18 & 19 & -2 & 15 \\ -1 & -53 & 27 & -28 & 2 & -24 \\ 2 & 52 & -27 & 29 & -4 & 21 \end{pmatrix}$$

whose REF is

$$\begin{pmatrix} 1 & 0 & -1/2 & 3/2 & -2 & -5/2 \\ 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system $(A+I)^2\boldsymbol{v}$ has $r_2 = 4$ linearly independent solutions. As $r_2 < r$, then we now consider $(A+I)^3\boldsymbol{v}$. Now

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -54 & 27 & -27 & 0 & -27 \\ 0 & 108 & -54 & 54 & 0 & 54 \\ 0 & 108 & -54 & 54 & 0 & 54 \\ 0 & -162 & 81 & -81 & 0 & -81 \\ 0 & 162 & -81 & 81 & 0 & 81 \end{pmatrix}$$

and it's easy to see (!) that the REF of this matrix is

$$\begin{pmatrix} 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $(A+I)^3\boldsymbol{v} = 0$ has $r_3 = 5$ linearly independent solutions, and as $r_3 = r$ we conclude this part of the proceedings. We calculate $s_1 = r_1 = 2$, $s_2 = r_2 - r_1 = 2$ and $s_3 = r_3 - r_2 = 1$; also $m_3 = s_3 = 1$, $m_2 = s_2 - s_3 = 1$ and $m_1 = s_1 - s_2 = 0$. Hence, associated to $\lambda = -1$, there is a 2 by 2 and a 3 by 3 Jordan block. As the other eigenvalue $\lambda = 2$ has multiplicity 1, then there's just a 1 by 1 Jordan block associated to $\lambda = 2$. Hence the Jordan canonical form of A is J=

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Let's compute the matrix P. We've already done most of the work for $\lambda = -1$. The Jordan blocks have sizes $N_1 = 3$ and $N_2 = 2$. We start by finding a vector $\boldsymbol{v}_{1,1}$ with $(A+I)^3\boldsymbol{v}_{1,1} = 0$ but $(A+I)^2\boldsymbol{v}_{1,1} \neq 0$. Looking at the REFs of these matrices we see that we can choose

$$\boldsymbol{v}_{1,1} = (1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)^t.$$

Now

$$\boldsymbol{v}_{1,2} = (A+I)\boldsymbol{v}_{1,1} = (1 \;\; 0 \;\; -3 \;\; -2 \;\; 1 \;\; -1)^t$$

and

$$\boldsymbol{v}_{1,3} = (A+I)\boldsymbol{v}_{1,2} = (1 \;\; -2 \;\; -1 \;\; 1 \;\; -1 \;\; 2)^t.$$

(As a check one verifies $(A+I)\boldsymbol{v}_{1,3} = 0$.) The next block is 2 by 2, so one must find $\boldsymbol{v}_{2,1}$ with $(A+I)^2\boldsymbol{v}_{2,1} = 0$, $(A+I)\boldsymbol{v}_{2,1} \neq 0$, and such that $\boldsymbol{v}_{2,1}$ is not linearly dependent on $\boldsymbol{v}_{1,1}$, $\boldsymbol{v}_{1,2}$ and $\boldsymbol{v}_{1,3}$. The vector

$$\boldsymbol{v}_{2,1} = (1 \;\; 1 \;\; 2 \;\; 0 \;\; 0 \;\; 0)^t$$

fits the bill, and

$$\boldsymbol{v}_{2,2} = (A+I)\boldsymbol{v}_{2,1} = (1 \;\; 1 \;\; -4 \;\; -2 \;\; 5 \;\; -4)^t.$$

Again one checks that $(A+I)\boldsymbol{v}_{2,2} = 0$ The matrix P_{-1} is the 6 by 5 matrix with columns $\boldsymbol{v}_{2,2}$, $\boldsymbol{v}_{2,1}$, $\boldsymbol{v}_{1,3}$, $\boldsymbol{v}_{1,2}$ and $\boldsymbol{v}_{1,1}$ in that order and so

$$P_{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 \\ -4 & 2 & -1 & -3 & 0 \\ -2 & 0 & 1 & -2 & 0 \\ 5 & 0 & -1 & 1 & 0 \\ -4 & 0 & 2 & -1 & 0 \end{pmatrix}.$$

One must now consider $\lambda = 2$. As this is a simple root, P_2 is just an eigenvector with eigenvalue 2. One such is

$$P_2 = (0 \;\; 1 \;\; -2 \;\; -2 \;\; 3 \;\; -3)^t$$

and sticking together P_{-1} and P_2 gives

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \\ -4 & 2 & -1 & -3 & 0 & -2 \\ -2 & 0 & 1 & -2 & 0 & -2 \\ 5 & 0 & -1 & 1 & 0 & 3 \\ -4 & 0 & 2 & -1 & 0 & -3 \end{pmatrix}.$$

One now checks that $P^{-1}AP = J$ as required!

RJC 25/1/95