Vectors and matrices: Problem sheet 4

*Please answer all questions, and **check** your answers whenever possible*
Solutions should be submitted before or on Thursday 2 December 2004

1. Give examples of matrices A and B where

   (a) AB exists but BA does not.

   (b) AB and BA both exist but have different sizes.

2. Find, **whenever possible**, the inverses of the following matrices:

$$\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 3 & -5 & -1 \\ -1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & -1 & 0 \\ 1 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 5 & 2 & 2 \end{pmatrix}.$$

   (In the second example, $i^2 = -1$.)

3. Let

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}.$$

   Find all matrices

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

   which satisfy $AX = XA$.

4. For each of the following matrices A, calculate A^2, A^3 and A^4. Guess (and if possible prove) a general formula for A^n:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

5. Let L be the line $y = x \tan\theta$ in the plane (so that L makes an angle of θ with the x-axis). Find the matrix which represents reflection through the line L. (Hint: where do the points $(1, 0)$ and $(0, 1)$ get reflected to?)

6. Find the equation of the image of the unit circle $x^2 + y^2 = 1$ under the transformation with matrix

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

7. Which of the following matrices are elementary? For each elementary matrix among them, write down its inverse.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. True or false? Justify your answers.

   (a) If A and B are invertible matrices of the same size then AB has inverse $A^{-1}B^{-1}$.

   (b) If A is a square matrix then $(I - A)^2 = I^2 - 2A + A^2$.

   (c) If A and B are square matrices of the same size then $A^2 - B^2 = (A + B)(A - B)$.

   (d) If A and B are square matrices of the same size then $(AB)^2 = A^2 B^2$.

   (e) If A and B are symmetric matrices of the same size then $A + B$ is symmetric.

   (f) If A and B are symmetric matrices of the same size then AB is symmetric.

9. An *orthogonal* matrix A is a square matrix which satisfies $AA^t = I$ (where A^t denotes the transpose of A). Prove that if A and B are orthogonal matrices of the same size then AB is also orthogonal.

   Verify that the 2-by-2 rotation matrices and reflection matrices (as in question 5) are orthogonal. Can you find any other orthogonal 2-by-2 matrices?

RJC 18/11/2004