

Vectors and matrices: Problem sheet 5

*Please answer all questions, and **check** your answers whenever possible
Solutions should be submitted before or on Thursday 13 January 2004*

1. Calculate the following determinants:

$$\begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 2 \\ 2 & 6 & 7 \\ 3 & 8 & 4 \end{vmatrix}, \quad \begin{vmatrix} 4 & 2 & 5 \\ 1 & 6 & -1 \\ 5 & 9 & 3 \end{vmatrix},$$

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 6 & 7 \\ 3 & 8 & 4 \end{vmatrix}, \quad \begin{vmatrix} 0 & 3 & 2 \\ 5 & 1 & 3 \\ 8 & 1 & 6 \end{vmatrix}, \quad \begin{vmatrix} 5 & 2 & 3 & 4 \\ 4 & 3 & 7 & 3 \\ 5 & 0 & 1 & 5 \\ 3 & 3 & 6 & 2 \end{vmatrix}.$$

2. Use row operations to calculate the following Vandermonde determinant:

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

3. Find factorizations of each of the following not-quite-Vandermonde determinants:

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}, \quad \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

4. Using row and column operations, or otherwise, find the following determinant:

$$\begin{vmatrix} 1 & t & 1 & 0 \\ 0 & 1 & t & 1 \\ 1 & 0 & 1 & t \\ t & 1 & 0 & 1 \end{vmatrix}.$$

5. Find the inverse of the following matrix both (i) by using Gaussian elimination, and (ii) by calculating the adjugate:

$$\begin{pmatrix} 0 & 6 & 7 \\ 3 & -2 & -3 \\ -1 & 7 & 8 \end{pmatrix}.$$

Which method do you consider more reliable? consider quicker? prefer?

6. Find the eigenvalues and eigenvectors of the following matrices, and determine which are diagonalizable:

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 \\ -2 & -6 \end{pmatrix}, \quad \begin{pmatrix} 20 & 29 \\ -10 & -14 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -4 & 4 \\ 1 & -1 & -1 \\ 1 & -4 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

7. For each of the following matrices A find a formula for A^n :

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & -3 & 2 \end{pmatrix}.$$

8. True or false? Justify your answers.

- (a) If A is an invertible matrix then $\det(A^{-1}) = \det(A)^{-1}$.
- (b) If A is a square matrix then $\det(-A) = -\det(A)$.
- (c) If A is an orthogonal matrix (for definition, see sheet 4) then $\det(A) = \pm 1$.