A new proof of some identities of Bressoud

Robin Chapman
School of Mathematical Sciences
University of Exeter
Exeter, EX4 4QE, UK
rjc@maths.ex.ac.uk
11 October 2001

Abstract

We provide a new proof of two identities due to Bressoud:

\[
\sum_{m=0}^{N} q^{m^2} \left[ \frac{N}{m} \right] = \sum_{m=\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \left[ \frac{2N}{N + 2m} \right]
\]

and

\[
\sum_{m=0}^{N} q^{m^2+m} \left[ \frac{N}{m} \right] = \frac{1}{1-q^{N+1}} \sum_{m=\infty}^{\infty} (-1)^m q^{m(5m+3)/2} \left[ \frac{2N + 2}{N + 2m + 2} \right]
\]

which can be considered as finite versions of the Rogers-Ramanujan identities.

MSC2000 classification: 05A19
In [1] Bressoud proves the following theorem, from which the Rogers-Ramanujan identities follow on letting \( N \to \infty \).

**Theorem 1** For each integer \( N \geq 0 \),
\[
\sum_{m=0}^{N} q^{m^2} \left[ \frac{N}{m} \right] = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \left[ \frac{2N}{N + 2m} \right] \tag{1}
\]
and
\[
\sum_{m=0}^{N} q^{m^2+m} \left[ \frac{N}{m} \right] = \frac{1}{1-q^{N+1}} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+3)/2} \left[ \frac{2N + 2}{N + 2m + 2} \right]. \tag{2}
\]

Here
\[
\left[ \frac{N}{m} \right] = \begin{cases} \frac{(q)_N}{(q)_m(q)_{N-m}} & \text{if } 0 \leq m \leq N; \\ 0 & \text{otherwise.} \end{cases}
\]

denotes a Gaussian binomial coefficient, where we adopt the standard \( q \)-series notation:
\[
(q)_n = \prod_{j=1}^{n} (1-q^j).
\]

We give an alternative proof of Theorem 1 by showing that the left and right sides of (1) and (2) satisfy the same recurrence relations.

Define, for integers \( a \) and \( N \geq 0 \),
\[
S_a(N) = \sum_{n=0}^{N} q^{n^2+an} \left[ \frac{N}{n} \right].
\]

**Lemma 1** For each integer \( N \geq 1 \) and each \( a \) we have
\[
S_a(N) = S_a(N - 1) + q^{N+a} S_{a+1}(N - 1) \tag{3}
\]
and
\[
S_a(N) = S_{a+1}(N - 1) + q^{a+1} S_{a+2}(N - 1). \tag{4}
\]

**Proof** Using the identity
\[
\left[ \frac{N}{n} \right] = q^{N-n} \left[ \frac{N-1}{n-1} \right] + \left[ \frac{N-1}{n} \right]
\]
gives

\[ S_a(N) = q^N \sum_{n=1}^{N} q^{n^2 + (a-1)n} \left[ \frac{N-1}{n-1} \right] + \sum_{n=0}^{N-1} q^{n^2 + an} \left[ \frac{N-1}{n} \right] \]

\[ = q^N \sum_{n=0}^{N-1} q^{(n+1)^2 + (a-1)(n+1)} \left[ \frac{N-1}{n} \right] + S_a(N-1) \]

\[ = q^{N+a} S_{a+1}(N-1) + S_a(N-1). \]

On the other hand, using the identity

\[ \left[ \begin{array}{c} N \\ n \end{array} \right] = \left[ \begin{array}{c} N-1 \\ n-1 \end{array} \right] + q^n \left[ \begin{array}{c} N-1 \\ n \end{array} \right] \]

gives

\[ S_a(N) = \sum_{n=1}^{N} q^{n^2 + an} \left[ \frac{N-1}{n-1} \right] + \sum_{n=0}^{N-1} q^{n^2 + (a+1)n} \left[ \frac{N-1}{n} \right] \]

\[ = \sum_{n=0}^{N-1} q^{(n+1)^2 + a(n+1)} \left[ \frac{N-1}{n} \right] + S_{a+1}(N-1) \]

\[ = q^{a+1} S_{a+2}(N-1) + S_{a+1}(N-1) \]

\[ \square \]

**Lemma 2** For integers \( N \geq 0 \) and each \( a \) we have

\[ S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1}S_{a+2}(N) = 0. \]

**Proof** Equating (3) and (4) gives

\[ S_a(N-1) + (q^{N+a} - 1)S_{a+1}(N-1) - q^{a+1}S_{a+2}(N-1) = 0 \]

for \( N \geq 1 \). Replacing \( N \) by \( N+1 \) gives

\[ S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1}S_{a+2}(N) = 0. \]

\[ \square \]

We shall use the \( a = 0 \) case of Lemma 2 which is

\[ S_0(N) + (q^{N+1} - 1)S_1(N) - qS_2(N) = 0. \] (5)

Clearly \( S_a(0) = 1 \) for all \( a \). Also for \( N > 0 \), (3) gives

\[ S_a(N) = S_0(N-1) + q^N S_1(N-1) \] (6)
and together with (5) gives
\[ S_1(N) = S_1(N-1) + q^{N+1}S_2(N-1) \]
\[ = S_1(N-1) + q^N [S_0(N-1) + (q^N - 1)S_1(N-1)] \]
\[ = q^N S_0(N-1) + (q^{2N} - q^N + 1)S_1(N-1). \] (7)

Together with the initial conditions \( S_0(0) = S_1(0) = 1 \), (6) and (7) completely define \( S_0(N) \) and \( S_1(N) \) for \( N \geq 0 \).

We now gather some consequences of these recurrences which will be used later.

Lemma 3 For \( N \geq 2 \) we have
\[ S_0(N) = (1 + q^{2N-1})S_0(N-1) + q^N(1 - q^N)S_1(N-2). \] (8)

and for \( N \geq 1 \) we have
\[ S_1(N) = q^NS_0(N) + (1 - q^N)S_1(N-1). \] (9)

Proof First of all from (6) and (7) we have
\[ S_1(N) - q^NS_0(N) = (1 - q^N)S_1(N-1) \]
and so for \( N \geq 2 \),
\[ S_1(N-1) - q^{N-1}S_0(N-1) = (1 - q^{N-1})S_1(N-1) \]
Hence by (6) again,
\[ S_0(N) = S_0(N-1) + q^NS_1(N-1) \]
\[ = S_0(N-1) + q^N [q^{N-1}S_0(N-1) + (1 - q^N)S_1(N-2)] \]
\[ = (1 + q^{2N-1})S_0(N-1) + q^N(1 - q^N)S_1(N-2) \]
and by also using (7),
\[ S_1(N) = q^NS_0(N-1) + (1 - q^N + q^{2N})S_1(N-1) \]
\[ = q^N [S_0(N) - q^NS_1(N-1)] + (1 - q^N + q^{2N})S_1(N-1) \]
\[ = q^NS_0(N) + (1 - q^N)S_1(N-1). \]

The recurrences (8) and (9) with the initial conditions \( S_0(0) = S_1(0) = 1 \), \( S_0(1) = 1 + q \) define \( S_0(N) \) and \( S_1(N) \) uniquely for \( N \geq 0 \).
Let 
\[ B_0(N) = \sum_m (-1)^m q^{m(5m+1)/2} \left[ \frac{2N}{N+2m} \right] \]
and 
\[ B_1(N) = \sum_m (-1)^m q^{m(5m+3)/2} \left[ \frac{2N+2}{N+2m+2} \right] \]
denote the sums appearing on the right sides of the identities in Theorem 1.

Setting \( r = N + 2m \) in the definition of \( B_0(N) \) gives
\[ B_0(N) = \sum_{r \equiv N (4)} q^{\frac{5}{8}(r-N)^2 + \frac{1}{8}(r-N)} \left[ \frac{2N}{r} \right] - \sum_{r \equiv N+2 (4)} q^{\frac{5}{8}(r-N)^2 + \frac{1}{8}(r-N)} \left[ \frac{2N}{r} \right]. \]

This suggests the notation
\[ A(M, k, b) = \sum_{2r \equiv M+k (8)} q^{\frac{5}{8}(r-M/2+b)^2} \left[ \frac{M}{r} \right] \]
so that
\[ q^{1/40} B_0(N) = A(2N, 0, 1/5) - A(2N, 4, 1/5). \]

Of course, \( A(M, k, b) = 0 \) if \( M + k \) is odd and \( A(M, k, b) \) depends only on \( M, b \) and the congruence class of \( k \) modulo 8. A similar computation yields
\[ q^{9/40} B_1(N) = A(2N + 2, 2, -2/5) - A(2N + 2, -2, -2/5). \]

We aim to show that \( B_0(N) \) and \( (1 - q^{N+1}) B_1(N) \) satisfy the same system of recurrences as \( S_0(N) \) and \( S_1(N) \).

**Lemma 4** We have
\[ A(M, k, b) = A(M, -k, -b) \]
for each \( M, k \) and \( b \).

**Proof** Replacing \( r \) by \( M - r \) in the sum for \( A(M, k, b) \) yields
\[ A(M, k, b) = \sum_{2r \equiv M+k (8)} q^{\frac{5}{8}(r-M/2+b)^2} \left[ \frac{M}{r} \right] \]
\[ = \sum_{2r \equiv M-k (8)} q^{\frac{5}{8}(r-M/2-b)^2} \left[ \frac{M}{r} \right] \]
\[ = A(M, -k, -b). \]

We now wish to produce recurrences for the \( A(M, k, b) \).
Lemma 5 We have

\[ A(M + 1, k, b) = A(M, k - 1, b + 1/2) + q^{M/2 + 1/10 - b} A(M, k + 1, b + 3/10) \]

and

\[ A(M + 1, k, b) = A(M, k + 1, b - 1/2) + q^{M/2 + 1/10 + b} A(M, k - 1, b - 3/10) \]

for each \( M, k \) and \( b \).

Proof Using the formula

\[ \left[ \begin{array}{c} M + 1 \\ r \end{array} \right] = \left[ \begin{array}{c} M \\ r - 1 \end{array} \right] + q^r \left[ \begin{array}{c} M \\ r \end{array} \right] \]

in the definition of \( A(M + 1, k, b) \) gives \( A(M + 1, k, b) = S_1 + S_2 \) where

\[
S_1 = \sum_{2r \equiv M + k + 1 \mod 2} q^\frac{r}{8}(r - M/2 - \frac{1}{2} + b)^2 \left[ \begin{array}{c} M \\ r - 1 \end{array} \right]
\]

\[
= \sum_{2s \equiv M + k - 1 \mod 2} q^\frac{s}{8}(s - M/2 + \frac{1}{2} + b)^2 \left[ \begin{array}{c} M \\ s \end{array} \right]
\]

\[
= A(M, k - 1, b + 1/2)
\]

and

\[
S_2 = \sum_{2r \equiv M + k + 1 \mod 2} q^{r + \frac{r}{8}(r - M/2 - \frac{1}{2} + b)^2} \left[ \begin{array}{c} M \\ r \end{array} \right].
\]

But

\[
r + \frac{5(r - M/2 - \frac{1}{2} + b)^2}{8} = \frac{5(r - M/2 + 3/10 + b)^2}{8} + \frac{M}{2} + \frac{1}{10} - b.
\]

Hence

\[ A(M + 1, k, b) = A(M, k - 1, b + 1/2) + q^{M/2 + 1/10 - b} A(M, k + 1, b + 3/10). \]

Consequently, by Lemma 4 also

\[
A(M + 1, k, b) = A(M + 1, -k, -b)
\]

\[
= A(M, -k - 1, -b + 1/2) + q^{M/2 + 1/10 + b} A(M, -k + 1, -b + 3/10)
\]

\[
= A(M, k + 1, b - 1/2) + q^{M/2 + 1/10 + b} A(M, k - 1, b - 3/10).
\]

\[ \square \]
It is convenient to note that replacing $M$ by $M - 1$ in these identities gives

$$A(M, k, b) = A(M - 1, k - 1, b + 1/2) + q^{M/2-2-5} A(M - 1, k + 1, b + 3/10)$$

$= A(M - 1, k + 1, b - 1/2) + q^{M/2-2+5} A(M - 1, k - 1, b - 3/10)$.

**Lemma 6** The sums $B_0(N)$ and $B_1(N)$ obey the recurrences

$$B_0(N) = (1 + q^{2N-1})B_0(N - 1) + q^N B_1(N - 2)$$

for $N \geq 2$ and

$$B_1(N) = (1 - q^{N+1})B_1(N - 1) + q^N (1 - q^{N+1})B_0(N)$$

for $N \geq 1$.

**Proof** We compute

$$A(2N, k, 1/5) = A(2N - 1, k + 1, -3/10) + q^{N-1/5} A(2N - 1, k + 1, -1/10)$$

$= A(2N - 2, k, 1/5) + q^{N-3/5} A(2N - 2, k + 2, 0) + q^{N-1} A(2N - 2, k - 2, 2/5) + q^{2N-1} A(2N - 2, k, 1/5)$

$= (1 + q^{2N-1})A(2N - 2, k, 1/5) + q^{N-3/5} A(2N - 2, k + 2, 0) + q^{N-1} A(2N - 2, k - 2, 2/5)$.

In particular

$$A(2N, 0, 1/5) = (1 + q^{2N-1})A(2N - 2, 0, 1/5) + q^{3/5} A(2N - 2, 2, 0) + q^{1/5} A(2N - 2, -2, 2/5).$$

and

$$A(2N, 4, 1/5) = (1 + q^{2N-1})A(2N - 2, 4, 1/5) + q^{3/5} A(2N - 2, 6, 0) + q^{1/5} A(2N - 2, 2, 2/5) + q^{3/5} A(2N - 2, -2, 0) + q^{1/5} A(2N - 2, 2, 2/5).$$

Noting that

$$A(2N - 2, 2, 0) = A(2N - 2, -2, 0)$$

and

$$A(2N - 2, 2, 2/5) = A(2N - 2, -2, -2/5)$$
substituting gives

\[ q^{1/40} B_0(N) = A(2N, 0, 1/5) - A(2N, 4, 1/5) \]
\[ = (1 + q^{2N-1})[A(2N - 2, 0, 1/5) - A(2N - 2, 4, 1/5)] \]
\[ + q^{N-1/5}[A(2N - 2, 2, -2/5) - A(2N - 2, -2, -2/5)] \]
\[ = (1 + q^{2N-1})q^{1/40} B_0(N - 1) + q^{N-1/5} q^{9/40} B_1(N - 2) \]

and so

\[ B_0(N) = (1 + q^{2N-1}) B_0(N - 1) + q^N B_1(N - 2). \]

Also

\[ A(2N + 2, k, -2/5) \]
\[ = A(2N + 1, k - 1, 1/10) + q^{N+1} A(2N + 1, k + 1, -1/10) \]
\[ = A(2N, k, -2/5) + q^{N+1/5} A(2N, k - 2, -1/5) \]
\[ + q^{N+1} A(2N, k, 2/5) + q^{2N+6/5} A(2N, k + 2, 1/5) \]
\[ = A(2N, k, -2/5) + q^{N+1} A(2N, -k, -2/5) \]
\[ + q^{N+1/5} A(2N, 2 - k, 1/5) + q^{2N+6/5} A(2N, k + 2, 1/5). \]

Consequently

\[ q^{9/40} B_1(N) = A(2N + 2, 2, -2/5) - (2N + 2, -2, -2/5) \]
\[ = A(2N, 2, -2/5) + q^{N+1} A(2N, -2 - 2/5) \]
\[ - A(2N, -2, -2/5) - q^{N+1} A(2N, 2, -2/5) \]
\[ + q^{N+1/5}[A(2N, 0, 1/5) - A(2N, 4, 1/5)] \]
\[ + q^{2N+6/5}[A(2N, 4, 1/5) - A(2N, 0, 1/5)] \]
\[ = (1 - q^{N+1})[q^{9/40} B_1(N - 1) + q^{N+1/5} q^{1/40} B_0(N)] \]

and so

\[ B_1(N) = (1 - q^{N+1}) B_1(N - 1) + q^N (1 - q^{N+1}) B_0(N). \]

By Lemma 3 \( S_0(N) \) and \( (1 - q^{N+1}) S_1(N) \) satisfy the same recurrences as \( B_0(N) \) and \( B_1(N) \). Also \( S_0(0) = 1 = B_0(0), \ S_0(1) = 1 + q = B_0(1) \) and \( (1 - q) S_1(0) = 1 - q = B_1(0) \). Consequently we deduce Theorem 1: \( S_0(N) = B_0(N) \) and \( (1 - q^{N+1}) S_1(N) = B_1(N) \).

**References**