Dynamic models for a long wave motion in a sheared pre-stressed incompressible elastic layer

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Motivation

1. Further understand influence of **pre-stress** on dispersion and stability;
2. Geo-physical applications of **shear deformation**.

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Small superimposed time-dependent motion

\[ \tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t). \]

Figure: Configurations of a pre-stressed body.
Simple shear primary deformation

\[ x_1 = X_1 + \epsilon X_2, \quad x_2 = X_2, \quad x_3 = X_3. \]

Figure: The *simple shear* deformation
Axes of deformation and natural axes

$$2\theta = \tan^{-1} \left( \frac{2}{\varepsilon} \right).$$

$$\lambda_1 = \cot \theta \equiv \lambda, \quad \lambda_2 = \tan \theta \equiv \lambda^{-1}, \quad \lambda_3 = 1.$$  

Figure: The angle $\theta$ between Eulerian coordinates $(x_1', x_2')$ and natural coordinate system of the layer $(x_1, x_2)$. 

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Main features of models

- Propagation of 2D waves in an incompressible elastic layer subject to primary simple shear deformation;
- Wave length considerably exceeds the layer thickness = long wave with small parameter scaled wave number;
- Long wave low and high frequency regimes;
  - Long-wave low-frequency $\eta \to 0$ and wave speed is finite $v \to const.$
  - Long-wave high-frequency $\eta \to 0$ and finite frequency $\Omega \to const.$
Equations of motion in layer axes for neo-Hookean material

Neo-Hookean strain energy function

\[ W = \frac{\mu}{2} \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right) - p(\lambda_1 \lambda_2 \lambda_3 - 1). \]

Equations of motion

\[-\lambda^2 p_{t,1} - \lambda p_{t,2} - \rho \lambda^2 \ddot{u}_1 - \rho \lambda \ddot{u}_2 + \mu \left( \lambda^4 + \lambda^2(p - 1) + 1 \right) (u_{1,11} + u_{2,11}) + \mu \lambda \left( \lambda^2(p - 2) + 2 \right) (u_{1,12} + u_{2,12}) + \mu \lambda^2 u_{1,22} + \mu \lambda \left( 1 + p \right) u_{2,22} = 0.\]

\[\lambda^2 p_{t,1} - \lambda^3 p_{t,2} + \rho \lambda^2 \ddot{u}_1 - \rho \lambda^3 \ddot{u}_2 - \mu \left( \lambda^4 + \lambda^2(p - 1) + 1 \right) (u_{1,11} + u_{2,11}) + \mu \lambda \left( \lambda^2(p - 2) + 2 \right) (u_{1,12} + u_{2,12}) - \mu \lambda^2 u_{1,22} + \mu \lambda^3 \left( 1 + p \right) u_{2,22} = 0.\]

Incompressibility condition

\[u_{1,1} + u_{2,2} = 0.\]
Harmonic wave solutions

Seek solutions of the form

\[(u_1, u_2, p_t) = (U_1, U_2, kP)e^{ikq_2}e^{ik(vt-x_1)}.\]

Equation for \(q\)

\[q^4 - 2 \epsilon q^3 + (2 + \epsilon^2 - \hat{v})q^2 - 2 \epsilon q + 1 + \epsilon^2 - \hat{v} = 0, \quad \hat{v} = \frac{\rho v^2}{\mu}.
\]

Solutions for \(q\)

\[q_1 = i, \quad q_2 = -i, \quad q_3 = \epsilon + i\kappa, \quad q_4 = \epsilon - i\kappa, \quad \kappa^2 = 1 - \hat{v}.
\]

Solutions for \(u_1, u_2, p_t, \tau_1\) and \(\tau_2\)

\[u_1 = \sum_{i=1}^{4} q_i A_i e^{ikq_i x_2}, \quad u_2 = \sum_{i=1}^{4} A_i e^{ikq_i x_2}, \quad p_t = k \sum_{i=1}^{4} P(q_i)A_i e^{ikq_i x_2},
\]

\[\tau_1 = C \sum_{i=1}^{4} f(q_i)A_i e^{ikq_i x_2}, \quad \tau_2 = C \sum_{i=1}^{4} g(q_i)A_i e^{ikq_i x_2}.
\]
Motivation to construct dynamic models

- **Analytical solutions of displacement and incremental pressure components considering several asymptotic orders of the problem;**
- **One equation** can be used instead of system of PDE to describe 2D motion.

**Asymptotic methodology**

Long wave low frequency motion in a layer with free faces
Dispersion relation

\[
q_0 \left( \rho^2 - \kappa^2 \right)^2 + q_0 \kappa^2 (q_0 + 2 \rho)^2 + 4 \kappa^2 \left( \rho^2 - \kappa^2 \right) (q_0 + 2 \rho) \sinh (\eta) \sinh (\eta \kappa)
\]

\[+ 2\kappa \left( pq_0 + \rho^2 + \kappa^2 \right)^2 \left( \cos (\eta \epsilon) - \cosh (\eta) \cosh (\eta \kappa) \right) = 0,
\]

\[q_0 = 1 + \epsilon^2 + \kappa^2, \quad \eta = kh.
\]

Numerical solution with \(\epsilon = 2, \rho = 0.5\)
Long wave low frequency approximations and asymptotic orders

Wave speed for two fundamental modes

Mode 1: \[ v_1 = v_1^{(0)} + \eta^2 v_1^{(2)} + O(\eta^4) = \]

\[ 1 - p^2 + \frac{\eta^2}{12}(\epsilon^2 + (p + 1)^2)p^2 + O(\eta^4), \]

Mode 2: \[ v_2 = v_2^{(0)} + \eta^2 v_2^{(2)} + O(\eta^4) = \]

\[ \epsilon^2 + 2p + 2 - \frac{\eta^2}{12}(\epsilon^2 + (p + 1)^2) + O(\eta^4). \]

Scalings for long wave low frequency non-dimensional governing equations

\[ u_1 \approx u_2 \approx p_t, \]

\[ u_1 = lu_1^*, \quad u_2 = lu_2^*, \quad p_t = \mu p_t^*, \quad x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\sqrt{\frac{\rho}{\mu}}. \]
Analytical solutions for displacements and incremental pressure

Form of solution

\[(u_1^*, u_2^*, p_t^*) = \sum_{l=0}^{m} (u_1^{(l)}, u_2^{(l)}, p_t^{(l)}) \eta^l + O(\eta^{m+1}).\]

Leading order problem

\[u_1^{(0)} = U^{(0,0)}(\xi, \tau), \quad u_2^{(0)} = V^{(0,0)}(\xi, \tau).\]

Second order problem

\[u_1^{(1)} = \{(-\lambda + \lambda^{-1}) U_{,\xi}^{(0,0)} - V_{,\xi}^{(0,0)} p\} \zeta + U_1^{(0,1)}, \quad u_2^{(1)} = -U_{,\xi}^{(0,0)} \zeta + U_2^{(0,1)},\]
\[p_t^{(0)} = -(p + 1) U_{,\xi}^{(0,0)} + (\lambda - \lambda^{-1}) V_{,\xi}^{(0,0)}.\]

Third order problem

\[u_1^{(2)} = \{(p + \lambda^2 - 2 + \lambda^{-2}) U_{,\xi\xi}^{(0,0)} + (\lambda - \lambda^{-1}) p V_{,\xi\xi}^{(0,0)}\} \zeta^2\]
\[+ \{(-\lambda + \lambda^{-1}) U_1^{(0,1)} - U_2^{(0,1)} p\} \zeta + U_1^{(0,2)},\]
\[u_2^{(2)} = \{(\lambda - \lambda^{-1}) U_{,\xi\xi}^{(0,0)} + p V_{,\xi\xi}^{(0,0)}\} \zeta^2 - \zeta U_1^{(0,1)} + U_2^{(0,2)},\]
\[p_t^{(1)} = \{(\lambda - \lambda^{-1}) p U_{,\xi\xi}^{(0,0)} + p (p + 1) V_{,\xi\xi}^{(0,0)}\} \zeta - (p + 1) U_1^{(0,1)} + (\lambda - \lambda^{-1}) U_2^{(0,1)}.\]
Governing equation

to define $U^{(0,0)}$ and $V^{(0,0)}$ and hence determine asymptotic solutions for $u_1, u_2, p_t$

$$\frac{\partial^2}{\partial \tau^2} \begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} + D \frac{\partial^2}{\partial \zeta^2} \begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} = 0.$$

where the matrix $D$ given by

$$D = \begin{pmatrix} -2(p + 1) & \frac{(\lambda^2-1)(1+p)}{\lambda} \\ \frac{(\lambda^2-1)(1+p)}{\lambda} & \frac{\lambda^2(1+p)-\lambda^4-1}{\lambda} \end{pmatrix}.$$

Eigenvalues of $D$ give leading order long wave phase speed limits.
Long wave high frequency motion in a layer with fixed faces
The dispersion relation

\[ 2 \kappa \left( \cosh(\eta) \cosh(\eta \kappa) - \cos(\eta \epsilon) \right) - \left( 1 + \epsilon^2 + \kappa^2 \right) \sinh(\eta) \sinh(\eta \kappa) = 0, \]
\[ \kappa = \sqrt{1 - \hat{v}}. \]

Numerical solution of the dispersion relation with \( \epsilon = 3, p = 2 \) indicates no fundamental modes
Long wave high frequency approximations and asymptotic orders

Two families of cut-off frequencies

\[ \Omega_1 = 2k\pi, \quad \tan(A) = A \quad \text{where} \quad A = \frac{\Omega_2}{2}, \]

Long wave high frequency third order approximations

\[ \Omega = \Omega_i^{(0)} + \Omega_i^{(2)} \eta^2 + \Omega_i^{(4)} \eta^4 + O(\eta^6), \quad i = 1, 2. \]

Relative asymptotic orders of displacements and pressure

first family of cut-off frequencies

\[ u_1 = u_1^*, \quad u_2 = \eta u_2^*, \quad p = \eta^{-1} p^*, \]

scalings \[ u_1 = l u_1^*, \quad u_2 = l \eta u_2^*, \quad p_t = \mu p_t^* \eta^{-1} = \mu p_t^* \eta^{-1}, \quad x_1 = l \xi, \quad x_2 = l \eta \zeta, \quad t = l \eta \sqrt{\frac{\rho}{\mu} \tau} = l \eta \sqrt{\frac{\rho}{\mu} \tau}, \]

second family of cut-off frequencies

\[ u_1 = u_1^*, \quad u_2 = \eta u_2^*, \quad p = \eta^{-2} p^*. \]

scalings \[ u_1 = l u_1^*, \quad u_2 = l \eta u_2^*, \quad p_t = \mu p_t^* \eta^{-2} = \mu p_t^* \eta^{-2}, \quad x_1 = l \xi, \quad x_2 = l \eta \zeta, \quad t = l \eta \sqrt{\frac{\rho}{\mu} \tau} = l \eta \sqrt{\frac{\rho}{\mu} \tau}. \]
Phenomenon of negative group velocity in dispersive curves

Numerical analysis: negative group velocity in the vicinity of cut-off frequencies for harmonics with even index

Analytical explanation: gradients in the vicinity of cut-off frequencies

\[ \Omega_1^{(2)} = -\frac{1 + 2 \epsilon^2}{4\pi n}, \quad \Omega_2^{(2)} = \frac{1 + 6 \epsilon^2}{6\Omega(0)}, \]
Analytical solutions for displacement components and incremental pressure- part 1

Cut-off frequencies $2k\pi$, main parameter in model long wave amplitude $U_{1s}^{(0,0)}$, general solution of order $m$

$$(u_1^*, u_2^*, p_t^*) = \sum_{m=0}^{r} (u_1^{(m)}, u_2^{(m)}, p_t^{(m)}) \eta^m + O(\eta^{r+1}),$$

Leading order problem:  
$$u_1^{(0)} = U_{1s}^{(0,0)} \sin(\Omega \zeta), \quad u_2^{(0)} = \frac{U_{1s,\xi}^{(0,0)}}{\Omega} \cos(\Omega \zeta) - \frac{U_{1s,\xi}^{(0,0)}}{\Omega},$$

Second order problem:  
$$u_1^{(1)} = U_{1c}^{(0,1)} \cos(\Omega \zeta) + U_{1s}^{(0,1)} \sin(\Omega \zeta) + U_{1s}^{(1,1)} \zeta \sin(\Omega \zeta) + v_1^{(0,1)},$$
$$u_2^{(1)} = U_{2c}^{(0,1)} \cos(\Omega \zeta) + U_{2c}^{(1,1)} \zeta \cos(\Omega \zeta) + v_2^{(1,1)} \zeta + v_2^{(0,1)}, \quad p_t^{(1)} = P_t^{(0,1)} + P_{t}^{(1,1)} \zeta,$$

Third order problem:
$$u_1^{(2)} = U_{1c}^{(0,2)} \cos(\Omega \zeta) + U_{1s}^{(0,2)} \sin(\Omega \zeta) + U_{1c}^{(1,2)} \zeta \cos(\Omega \zeta) + U_{1s}^{(1,2)} \zeta \sin(\Omega \zeta) + U_{1s}^{(2,2)} \zeta^2 \sin(\Omega \zeta) + v_1^{(0,2)} + v_1^{(1,2)} \zeta,$$
$$u_2^{(2)} = U_{2c}^{(0,2)} \cos(\Omega \zeta) + U_{2c}^{(0,2)} \sin(\Omega \zeta) + U_{2c}^{(1,2)} \zeta \cos(\Omega \zeta) + U_{2s}^{(1,2)} \zeta \sin(\Omega \zeta) + U_{2c}^{(2,2)} \zeta^2 \cos(\Omega \zeta)$$
$$+ v_2^{(0,2)} + v_2^{(1,2)} \zeta + v_2^{(2,2)} \zeta^2, \quad p_t^{(2)} = P_t^{(0,2)} + P_{t}^{(1,2)} \zeta + P_{t}^{(2,2)} \zeta^2.$$
Analytical solutions for displacement components and incremental pressure- part 2

Cut-off frequencies \( \tan(A) = A = \frac{\Omega^2}{2} \), main parameter in model pressure increment \( P_t^{(0,0)} \)

Leading order problem: \( u_1^{(0)} = -\frac{P_{t,\xi}^{(0,0)} \cos(\Omega \zeta)}{\Omega^2} + \frac{P_{t,\xi}^{(0,0)} \sin(\Omega \zeta)}{2\Omega} + \frac{P_{t,\xi}^{(0,0)}}{\Omega^2}, \)

\[ u_2^{(0)} = \frac{P_{t,\xi,\xi}^{(0,0)} \cos(\Omega \zeta)}{2\Omega^2} + \frac{P_{t,\xi,\xi}^{(0,0)} \sin(\Omega \zeta)}{3\Omega^3} - \frac{P_{t,\xi,\xi}^{(0,0)}}{2\Omega^2} - \frac{P_{t,\xi,\xi}^{(0,0)}}{2\Omega^2}, \quad p_t^{(0)} = P_t^{(0,0)} (\xi, \tau). \]

Second order problem: \( u_1^{(1)} = U_{1c}^{(0,1)} \cos(\Omega \zeta) + U_{1s}^{(0,1)} \sin(\Omega \zeta) + U_{1c}^{(1,1)} \zeta \cos(\Omega \zeta) + U_{1s}^{(1,1)} \zeta \sin(\Omega \zeta) + v_1^{(0,1)}, \)

\[ u_2^{(1)} = U_{2c}^{(0,1)} \cos(\Omega \zeta) + U_{2s}^{(0,1)} \sin(\Omega \zeta) + U_{2c}^{(1,1)} \zeta \cos(\Omega \zeta) + U_{2s}^{(1,1)} \zeta \sin(\Omega \zeta) + v_2^{(0,1)} + v_2^{(1,1)} \zeta, \quad p_t^{(1)} = P_t^{(0,1)} (\xi, \tau), \]

Third order problem:

\[ u_1^{(2)} = U_{1c}^{(0,2)} \cos(\Omega \zeta) + U_{1s}^{(0,2)} \sin(\Omega \zeta) + U_{1c}^{(1,2)} \zeta \cos(\Omega \zeta) + U_{1s}^{(1,2)} \zeta \sin(\Omega \zeta), \]

\[ + U_{1c}^{(2,2)} \zeta^2 \cos(\Omega \zeta) + U_{1s}^{(2,2)} \zeta^2 \sin(\Omega \zeta) + v_1^{(0,2)} + v_1^{(1,2)} \zeta + v_1^{(2,2)} \zeta^2, \]

\[ u_2^{(2)} = U_{2c}^{(0,2)} \cos(\Omega \zeta) + U_{2s}^{(0,2)} \sin(\Omega \zeta) + U_{2c}^{(1,2)} \zeta \cos(\Omega \zeta) + U_{2s}^{(1,2)} \zeta \sin(\Omega \zeta), \]

\[ + U_{2c}^{(2,2)} \zeta^2 \cos(\Omega \zeta) + U_{2s}^{(2,2)} \zeta^2 \sin(\Omega \zeta) + v_2^{(0,2)} + v_2^{(1,2)} \zeta + v_2^{(2,2)} \zeta^2 + v_2^{(3,2)} \zeta^3, \]

\[ p_t^{(2)} = P_t^{(0,2)} + P_t^{(1,2)} \zeta + P_t^{(2,2)} \zeta^2. \]
1D governing equations aim to describe 2D motion

Cut-off frequencies $2k \pi$:

$$\frac{\rho h^2}{\lambda^2 \gamma} \frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 t^2} + 4k^2 \pi^2 U_{1s}^{(0,0)} + h^2 \left(\frac{2 + 2 \lambda^4 - 3 \lambda^2}{\lambda^2}\right) \frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 x_1^2} = 0,$$

elliptic type:

$$B_{e}^{(1)} = \left(\frac{2 + 2 \lambda^4 - 3 \lambda^2}{\lambda^2}\right) > 0.$$

Cut-off frequencies $\tan(A) = A$ $A = \frac{\Omega_2}{2}$:

$$\frac{\rho h^2}{\lambda^2 \gamma} \frac{\partial^2 P_t^{(0,0)}}{\partial^2 t^2} + \Omega^2 P_t^{(0,0)} - h^2 \left(\frac{6 \lambda^4 + 6 - 11 \lambda^2}{3 \lambda^2}\right) \frac{\partial^2 P_t^{(0,0)}}{\partial^2 x_1^2} = 0,$$

hyperbolic type:

$$B_{e}^{(2)} = \left(\frac{6 \lambda^4 + 6 - 11 \lambda^2}{3 \lambda^2}\right) > 0.$$

both coincide with previous results provided: $\lambda = 1$, $\alpha = \beta = \gamma$. 
Summary

- Equations of motions were obtained;
- Dispersion relations was derived and analyzed for different boundary value problems;
- Relative asymptotic orders of displacements and incremental pressure was established to construct dynamic model;
- Simplified dynamic model was presented for long wave low frequency 2D motion in a layer with free faces;
- 1D dynamic model can was presented for long wave high frequency 2D motion;
- Analytical solutions for displacement components and incremental pressure were obtained till third order;
- Phenomenon of negative group velocity was investigated.
conferences

- Euromech Colloquium 481 (2007) UK: Edge and surface waves;

papers