Non-obvious Thresholds for Rate-Induced Bifurcations in Slow-Fast Systems.

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Rate-induced bifurcations occur in forced systems where there is a stable state for every fixed level of forcing. When the forcing varies too fast, the system fails to adiabatically follow the continuously changing stable state and destabilises. This instability defines often non-obvious thresholds at which real-world systems fail to adapt to changing external conditions. We report on a novel threshold type whose intricate band structure is organised by composite canard trajectories due to a folded saddle-node singularity. The results are obtained for slow-fast dynamical systems with one fast and two slow variables, using modern concepts from geometric singular perturbation theory.

This paper studies instabilities which describe the failure of a physical system to adapt to changing external conditions. They are referred to as rate-induced bifurcations [1, 2], and are characterised by critical rates of external forcing [1, 2] and non-obvious thresholds [1, 3].

Conceptually, a rate-induced bifurcation is a dissipative analogue of the transition between an adiabatic and a non-adiabatic process [4]. It occurs in systems with a (unique) stable state that exists continuously for all fixed values of the external input [Fig. 1(a)–(b)]. When the external input varies in time, the position of the stable state changes and the system tries to keep pace with the changes. The forced system adiabatically follows or tracks the continuously changing stable state if the external input varies slowly enough [Fig. 1(a)]. However, there are systems that fail to track the changing stable state if the external input varies too fast. These systems have initial states that destabilise above some critical rate of forcing [Fig. 1(b)]. An instability occurs, even though there is no obvious loss of stability. Moreover, there may be no obvious threshold separating the two types of behaviour in Fig. 1(b). This is in contrast to dynamic bifurcations [5], which can be explained by classical bifurcations of the stable state at some critical level of external input [Fig. 1(c)]. The forced system destabilises around this critical level, independent of the initial state and of the rate of change.

In climate science and ecology one speaks of “rate-induced tipping points” [1, 6] and the “critical rate hypothesis” [7], respectively, to describe sudden changes in the state of the system when external conditions change too fast (e.g. dry and hot climate anomalies or wet periods). In neuroscience, type III excitable nerves [8, Ch. 7] accommodate slow changes in an externally applied voltage, but an excitation occurs if the voltage increases too fast [9, 10]. In the absence of an obvious threshold, scientists are often puzzled by the actual boundary separating initial states that adapt to changing external conditions from those that fail to adapt.

Thresholds for rate-induced bifurcations still remain fairly unexplored because they cannot, in general, be captured by traditional bifurcation theory or an asymptotic approach. The first non-obvious threshold was identified only recently, in the context of excitability, as a folded saddle canard in slow-fast systems [1]. This finding explained a rate-induced climate tipping point termed the “compost-bomb instability”—a sudden release of soil carbon from peat lands into the atmosphere above some critical rate of global warming, which puzzled climate carbon-cycle scientists [1, 11]. Subsequently, folded saddle canards were identified as non-obvious “firing thresholds” for type III neurons [3, 12].

Here, we reveal a novel threshold type with an intricate band structure. The threshold is identified with new composite canards, and arises from the complicated dynamics near a type I folded saddle-node singularity [13, 14]. The results follow from derivation of necessary and sufficient conditions for the existence of non-obvious thresholds.

The general framework is developed for externally forced slow-fast dynamical systems akin to simple climate and neuron models [1, 3, 11, 12, 15, 16]. Specifically, we consider

\[ \delta \frac{dx}{dt} = f(x, y, \lambda(\epsilon t), \delta), \]  
\[ \frac{dy}{dt} = g(x, y, \lambda(\epsilon t), \delta), \]

with a fast variable \( x \), slow variable \( y \), sufficiently smooth functions \( f \) and \( g \), and a small parameter \( 0 < \delta \ll 1 \). The time-varying external input \( \lambda(\epsilon t) \) evolves on a slow time scale \( \tau = \epsilon t \), and remains between \( \lambda_{\min} \) and \( \lambda_{\max} \).

When \( \lambda \) does not vary in time, i.e. when \( \epsilon = 0 \), Eqs. (1)–(2) define a dynamical system with one fast...
FIG. 2: (Color online) (a)–(b) Solutions starting at two different initial states (dots) on $S^a$, near the changing stable state $\tilde{x}$. (c) Initial states on $S^a$ that (red) destabilise or (blue) track $\tilde{x}(\lambda(\tau))$. Panels (a) for $\epsilon = 0.06$ and (b)–(c) for $\epsilon = 0.216$ show the behaviour below and above the critical rate, respectively. We used Eqs. (1)–(2) with $\delta = 0.01$, $f$ and $g$ given by Eq. (12), $\lambda(\tau)$ given by Eq. (13), $\lambda_{\text{max}} = 2.5$, and (a)–(b) $\lambda(0) = -1.7$. The critical manifold $S(\lambda)$ is given by $y = -\lambda - x(x - 1)$, has a fold $F(\lambda)$ at $(x, y) = \left(\frac{1}{2}, -\lambda + \frac{1}{4}\right)$ and a unique stable steady state $\tilde{x}(\lambda)$ at $(x, y) = (0, -\lambda)$. The $\lambda$ axis can be transformed into a slow time axis using Eq. (13).

and one slow variable, and a parameter $\lambda$. In the singular limit $\delta = 0$, the slow subsystem $dy/dt = g(x, y, \lambda, 0)$ evolves on the one-dimensional critical manifold $S(\lambda)$, defined by $f(x, y, \lambda, 0) = 0$. Alternatively, $S(\lambda)$ consists of steady states of the fast subsystem $dx/dT = f(x, y, \lambda, 0)$, where $T = t/\delta$ is the fast time scale and $y$ acts as a second parameter. The critical manifold can have an attracting part $S^r(\lambda)$ and a repelling part $S^s(\lambda)$, which are separated by a fold point $F(\lambda)$ tangent to the fast $x$ direction. To make precise statements about non-obvious thresholds for rate-induced bifurcations we assume for every fixed $\lambda$ between $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ (Fig. 2):

(A1) The critical manifold $S(\lambda)$ is locally a graph over $x$ with a single fold $F(\lambda)$ tangent to the fast $x$ direction, defined by $\partial f/\partial x = 0$ and $\partial^2 f/\partial x^2 \neq 0$.

(A2) Near $F(\lambda)$, $S^s(\lambda)$ contains just one steady state $\tilde{x}(\lambda)$ which is asymptotically stable and varies continuously with $\lambda$.

For $0 < \delta \ll 1$, where the steady states $S(\lambda)$ of the fast subsystem are hyperbolic (e.g. away from $F$), system (1)–(2) has an invariant slow manifold $S_\delta(\lambda)$ which lies close to $S(\lambda)$ and has the same stability type as $S(\lambda)$ [17, 18].

When $\lambda$ varies smoothly in time such that $0 < \epsilon \lesssim 1$ and $0 < \delta \ll \epsilon$, Eqs. (1)–(2) define a dynamical system with one fast and two slow variables

$$\delta \epsilon \frac{dx}{d\tau} = f(x, y, \lambda(\tau), \delta),$$  
$$\epsilon \frac{dy}{d\tau} = g(x, y, \lambda(\tau), \delta),$$  
$$\frac{d\lambda}{d\tau} = 1.$$  

Then the critical manifold $S$ and the slow manifold $S_\delta$ are two-dimensional, and $\tilde{x}$ and $F$ form curves (Fig. 2). When $\lambda(\tau)$ varies slowly enough, the forced system (1)–(2) tracks the continuously changing stable state $\tilde{x}(\lambda(\tau))$. However, sometimes it may fail to track. For a given initial state, we say that system (1)–(2) destabilises if the trajectory crosses $F$ and moves away from $\tilde{x}$ along the fast $x$ direction. We define the critical rate as the smallest $\epsilon$ above which there are initial states in $S_\delta^0$ that destabilise. Then we define the instability threshold as the boundary within $S_\delta^0$ separating initial states that track $\tilde{x}(\lambda(\tau))$ from those that destabilise.

Figure 2(a)–(b) shows two trajectories of Eqs. (1)–(2) for different initial states on $S^a$. Below the critical rate, all trajectories track, and eventually converge to $\tilde{x}(\lambda(\tau))$ [Fig. 2(a)]. However, above the critical rate there are initial states near $\tilde{x}$ that fail to track $\tilde{x}(\lambda(\tau))$, and destabilise [Fig. 2(b)]. The two qualitatively different behaviours in Fig. 2(b) show there exists a threshold within $S_\delta^a$. What is more, it has an intriguing band structure, that has not been reported to date [Fig. 2(c)]. However, it is not obvious what defines such threshold structures.

We set $\delta = 0$ to identify the dynamical mechanism for non-obvious thresholds. System (3)–(5) is reduced to the slow dynamics on $S$, and projected onto the $(x, \lambda)$-plane by differentiating Eq. (3) with respect to slow time $\tau$:

$$\frac{dx}{d\tau} = \left.\frac{g \partial f/\partial y + (\partial f/\partial \lambda)(d\lambda/d\tau)}{\epsilon \partial f/\partial x}\right|_{S^s},$$  
$$\frac{d\lambda}{d\tau} = 1.$$  

It now becomes clear that if a trajectory deviates too much from $\tilde{x}$ and approaches a typical point on $F$, then $\partial f/\partial x$ in Eq. (6) approaches zero, and $x$ diverges off to infinity in finite slow time $\tau$. However, there may be special points on $F$ where

$$[g \partial f/\partial y + (\partial f/\partial \lambda)(d\lambda/d\tau)]|_F = 0,$$

so $dx/d\tau$ remains finite. Such special points are referred to as folded singularities [19, 20]. The corresponding trajectories, that cross from $S^a$ along the eigendirections of a folded singularity onto $S^r$, are referred to as singular canards [19]. To study the flow near $F$, where Eqs. (6)–(7) are singular, we reverse time on $S^r$ according to [23]:

$$d\tau = -d\epsilon \cdot (\partial f/\partial x)|_S.$$
and the folded saddle singular canard $F$ and singularity. If a folded saddle is the only folded threshold if there is a folded saddle singularity within (1)–(2) with assumptions (A1)–(A2) has an instability theorem (8), and the non-zero speed condition $\delta t$ at the same time $t$. Then (1)–(2) has a critical rate $\epsilon_c$. Existence of non-obvious thresholds. The forced system (1)–(2) with assumptions (A1)–(A2) has an instability threshold if there is a folded saddle singularity within $(\lambda_{\text{min}}, \lambda_{\text{max}})$. The system is guaranteed to have an instability threshold if a folded saddle is the only folded singularity. Existence of non-obvious thresholds. The forced system (1)–(2) with assumptions (A1)–(A2) has an instability threshold if there is a folded saddle singularity within $(\lambda_{\text{min}}, \lambda_{\text{max}})$. The system is guaranteed to have an instability threshold if a folded saddle is the only folded singularity.
FIG. 4: (Color online) (a) Initial states on the critical manifold $S$ that (white) destabilise or (gray) track $\tilde{x}(\lambda(\epsilon t))$ for Eqs. (1)–(2) and (12)–(13) with $\delta = 0.01$ and $\epsilon = 0.204$. Inset shows gray band between $c$ and $d$; a similar band exists between $e$ and $f$. Labels $b$–$g$ for $\lambda(0) = -0.7$ denote different threshold components including: (b) the folded saddle maximal canard $\gamma^S$, (c) the strong folded node maximal canard $\gamma^N$, (d) a composite canard that follows $\gamma^S_0$ and $\gamma^S$, (e) a secondary folded node maximal canard, (f) a composite canard that follows a secondary maximal canard and $\gamma^S$, (g) a secondary folded node maximal canard.

Figure 4 identifies different components of the complicated threshold. They consist of known maximal canards such as (b) $\gamma^S$, (c) $\gamma^N$, and [(e) and (g)] secondary folded node maximal canards that bifurcate off $\gamma^S$ [21]. Most interestingly, we uncover special composite canards, which first (d) follow $\gamma^N$, or (f) follow a secondary folded node maximal canard, and then [(d) and (f)] follow $\gamma^S$. This explains the intriguing band structure in Figs. 2(c) and 3(c).

**Case 2: Simple threshold due to an isolated folded saddle singularity.** Consider example (12) subject to an exponential approach at a rate $\epsilon$:

$$\lambda(\epsilon t) = \lambda_{\text{max}} \left(1 - c e^{-\epsilon t}\right),$$

where $\lambda \in (0, \lambda_{\text{max}})$ and $c = 1 - \lambda(0)/\lambda_{\text{max}}$. It follows from Eqs. (9)–(12) and (14) that, upon increasing $\epsilon$, an isolated folded saddle $FS$ at $(x, \lambda) = (1/2, \lambda_{\text{max}} - (2\epsilon)^{-1})$ enters $(0, \lambda_{\text{max}})$ via its lower boundary at $\epsilon = (2\lambda_{\text{max}})^{-1}$, which is approximately the critical rate $\epsilon_c$ for $0 < \delta \ll 1$. Hence, there is an instability threshold given by the folded saddle maximal canard $\gamma^S$ [Fig. 5] [1, 3]. Note, this threshold is very similar to that in Fig. 3(d).

In summary, we described non-obvious thresholds for rate-induced bifurcations, where a forced system fails to track a continuously changing stable state above some critical rate of forcing. These thresholds are “non-obvious” because they cannot be revealed by traditional bifurcation theory and, in spite of their cross-disciplinary nature, still remain largely unexplored. We used concepts from geometric singular perturbation theory, namely folded singularities and canard trajectories, to study the existence of rate-induced bifurcations with non-obvious thresholds in a class of slow-fast dynamical systems. The general analysis led us to a novel threshold type that has an intriguing band structure. The structure contains a new kind of canard, referred to as a composite canard. It arises from an interplay of the complicated dynamics due to a folded node singularity, and the threshold behaviour due to a folded saddle singularity. Non-obvious thresholds are important in the real world because they separate initial states that adapt to changing external conditions from those that fail to adapt, and destabilise. For example, they show where climate or ecosystem fails to adapt to a rapidly changing environment [1, 2, 7, 11, 25, 26], and where type III excitable cells “fire” if the voltage stimulus rises fast enough [3, 8–10]. More generally, our results about non-obvious thresholds for rate-induced bifurcations give new insight into non-adiabatic processes in dissipative sys-
tems, and should be relevant to a wide range of problems.

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