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Convection Currents in a Porous Medium

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The problem is considered of the convection of a fluid through a permeable medium as the result of a vertical temperature-gradient, the medium being in the shape of a flat layer bounded above and below by perfectly conducting media. It appears that the minimum temperature-gradient for which convection can occur is approximately $4\pi^2h^2\mu/kg\rho_0\alpha D^2$, where h^2 is the thermal diffusivity, g is the acceleration of gravity, μ is the viscosity, k is the permeability, α is the coefficient of cubical expansion, ρ_0 is the density at zero temperature, and D is the thickness of the layer; this exceeds the limiting gradient found by Rayleigh for a simple fluid by a factor of $16D^2/27\pi^2k\rho_0$. A numerical computation of this gradient, based upon the data now available, indicates that convection currents should not occur in such a geological formation as the Woodbine sand of East Texas (west of the Mexia Fault zone); in view of the fact, however, that the distribution of NaCl in this formation seems to require the existence of convection currents, and in view of the approximations involved in applying the present theory, it seems safe tentatively, to conclude that convection currents do exist in this formation and that the expression given above predicts excessive minimum gradients when applied to such a formation.

I. INTRODUCTION

The problem of the occurrence of convection-currents in a fluid which is heated from below, has been examined by Rayleigh,[1] Jeffreys,[2,3,4] and Low.[5] The solution of the problem involves the finding of the relationship (between the physical parameters of the medium) which, so to speak, separates stability from instability.

Rayleigh solved the problem for a set of boundary conditions that are somewhat artificial; but the work of Jeffreys shows that the introduction of a more nearly realistic set of boundary conditions does not change the limiting relationship by more than a factor of three. The analysis of this paper follows that of Rayleigh, since his method affords a clearer insight into the physical problem.

Refer all measurements to Cartesian coordinates, of which the z-axis is directed vertically upward; let ρ and μ represent the density and viscosity of the fluid respectively, and let k represent the permeability of the porous medium. Let p represent the pressure in the fluid; and let u, v, and w represent the x-, y-, and z-components respectively of the velocity of the fluid. Let g represent the acceleration of gravity. Then, as Muskat has shown,[6] the hydrodynamic equations of motion of the fluid in the porous medium are

$$\left.\begin{aligned} u&=-(k/\mu)(\partial p/\partial x),\\ v&=-(k/\mu)(\partial p/\partial y),\\ w&=-(k/\mu)(\partial p/\partial z+g\rho),\end{aligned}\right\} \quad (1)$$

if the only external force is gravity. It will be supposed that the density, ρ, varies only with temperature and according to the relation

$$\rho=\rho_0(1-\alpha\Theta), \quad (2)$$

where Θ is the temperature and ρ_0 is the density at $\Theta=0$.

Other relationships which must be considered simultaneously with Eqs. (1) and (2) are those of continuity and of thermal conduction. If the fluid is supposed to be incompressible, the equation of continuity becomes

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0. \quad (3)$$

If h^2 is the thermal diffusivity, the equation for the conduction of heat is

$$D(\Theta)/Dt=h^2\nabla^2\Theta, \quad (4)$$

in which the operator

$$D/Dt=\partial/\partial t+u(\partial/\partial x)+v(\partial/\partial y)+w(\partial/\partial z);$$

t, of course, represents time.

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II. SOLUTION OF THE PROBLEM

To prepare Eqs. (1)–(4) for solution, first separate Θ into two parts: a steady-state temperature, βz, due to the gradient β; and the deviation, θ, therefrom. That is, let

$$\Theta=\beta z+\theta. \tag{5}$$

Secondly, introduce the potential-function

$$\Omega=p+g\rho_0 z-g\rho_0\alpha\int\beta z dz. \tag{6}$$

Finally, suppose that u, v, w, and θ and their rates of change are small. Then by Eqs. (5) and (6), Eqs. (1) become

$$\left.\begin{aligned} u&=-(k/\mu)(\partial\Omega/\partial x),\\ v&=-(k/\mu)(\partial\Omega/\partial y),\\ w&=-(k/\mu)(\partial\Omega/\partial z)+\Lambda\theta,\end{aligned}\right\} \tag{7}$$

where

$$\Lambda=kg\rho_0\alpha/\mu. \tag{8}$$

Again, Eq. (5) with Eq. (4) yields

$$D\theta/Dt+w\beta=h^2\nabla^2\theta;$$

and by the assumption of smallness as stated above, $D\theta/Dt=\partial\theta/\partial t$, so that this last equation becomes

$$\partial\theta/\partial t+w\beta=h^2\nabla^2\theta. \tag{9}$$

Let us suppose that u, v, w, and θ, and also Ω, are each proportional to

$$e^{ilx}e^{imy}e^{nt}; \tag{10}$$

then Eqs. (7) become

$$\left.\begin{aligned} u&=-i(kl/\mu)\Omega,\\ v&=-i(km/\mu)\Omega,\\ w&=-(k/\mu)(\partial\Omega/\partial z)+\Lambda\theta,\end{aligned}\right\} \tag{11}$$

while the equation of continuity, Eq. (3), becomes

$$ilu+imv+\partial w/\partial z=0. \tag{12}$$

Note that Eqs. (11) and (12) together give

$$\Omega=-(1/(l^2+m^2))(\mu/k)(\partial w/\partial z); \tag{13}$$

and that this Eq. (13) together with the last of Eqs. (11) gives

$$w=(1/(l^2+m^2))(\partial^2 w/\partial z^2)+\Lambda\theta. \tag{14}$$

The equation for the conduction of heat, Eq. (9), thus becomes

$$n\theta+w\beta=h^2(\partial^2/\partial z^2-l^2-m^2)\theta. \tag{15}$$

The treatment of the analogous problem for a simple fluid by Jeffreys is based upon the fact that the stable condition is distinguished from the unstable condition by the mathematical condition that $n=0$. Then w is eliminated from Eqs. (14) and (15). The resulting fourth-order equation in θ (sixth-order for the problem of the simple fluid) is solved subject to the boundary-conditions discussed by Jeffreys in references (2) and (3). These boundary-conditions yield a sequence of characteristic (eigen-) numbers, the smallest of which specifies the relation between the physical parameters for critical instability.

To continue with the solution of the present problem, note that if the fluid has free surfaces at both top and bottom, and if the media above and below the fluid are perfect conductors, then both w and θ will be zero at both the bottom ($z=0$) and the top ($z=D$) of the fluid layer. A reasonable assumption, then, is that w and θ are proportional to $\sin sz$, where $s=r\pi/D$ and r is an integer. If this assumption be adopted, Eq. (14) becomes

$$w=-(s^2/(l^2+m^2))w+\Lambda\theta, \tag{16}$$

and Eq. (15) becomes

$$n\theta+w\beta=-h^2(l^2+m^2+s^2)\theta. \tag{17}$$

Eqs. (16) and (17) will be consistent if and only if

$$n=[-\beta\Lambda(\sigma-s^2)-h^2\sigma^2]/\sigma, \tag{18}$$

where

$$\sigma=l^2+m^2+s^2. \tag{19}$$

Eq. (18) indicates that if β is positive, n is negative, so that any convective disturbance will decrease in amount exponentially with time; i.e., convection currents cannot occur. On the other hand, if the fluid is heated from below so that β is negative, say $-\beta'$, n can be positive and convection currents may develop.

If $\beta=-\beta'$, $\beta'>0$, n will be positive if

$$\beta' g\alpha(\sigma-s^2)>h^2\mu\sigma^2/k\rho_0. \tag{20}$$

This is similar to the condition for positive n,

$$\beta' g\alpha(\sigma-s^2)>h^2\mu\sigma^3, \tag{21}$$

derived by Rayleigh for a simple fluid.

To complete the solution of the present problem, note that Eq. (18) indicates that if l^2+m^2 is very small or very large, n is negative; that is, the value of σ for which n is just zero is the value for which the numerator has a maximum with respect to σ. Thus the critical condition between stability and instability ($n=0$) is characterized by

$$\beta'\Lambda(\sigma-s^2)-h^2\sigma^2=0 \tag{22}$$

and

$$\beta'\Lambda-2h^2\sigma=0; \tag{23}$$

the Eq. (23) is the derivative of Eq. (22) with respect to σ. These two equations require that

$$\left.\begin{array}{c}\sigma=2s^2\\ l^2+m^2=s^2.\end{array}\right\} \text{and} \quad (24)$$

For a given s, therefore, the necessary condition for the formation of convection currents is (by Eqs. (22) and (24))

$$\beta'\Lambda > h^2\sigma^2/s^2 = 4h^2s^2. \quad (25)$$

Finally, the minimum value of $s=r\pi/D$ is π/D; so from Eq. (25), the result appears that convection currents may develop only for temperature-gradients

$$\beta' > 4\pi^2h^2\mu/kg\rho_0\alpha D^2. \quad (26)$$

This criterion, Eq. (26), for the formation of convection currents in a fluid in a porous medium may be compared with the equivalent condition developed by Rayleigh for the formation of convection currents in a simple fluid:

$$\beta' > 27\pi^4h^2\mu/4g\alpha D^4. \quad (27)$$

The minimum temperature-gradient necessary for the formation of convection currents in a porous medium is greater than that required for convection currents in the free fluid by the factor

$$(16/27)(D^2/\pi^2k\rho_0). \quad (28)$$

It should be noted that this ratio varies as D^2, where D is the thickness of the fluid layer, and so may become quite large.

III. AN APPLICATION OF GEOPHYSICAL INTEREST; DISCUSSION

An interesting application of this theory can be made to the problem of a subterranean inclined sand-layer containing a fluid, which layer is bounded above and below by impervious layers. The accuracy is limited, in this application, by conceivable inaccuracies in Eq. (26) (which has not been subjected to explicit experimental tests, and which has not been refined by analyses of the types of Jeffreys and of Low), by the simplifying approximations used in deriving Eq. (26), and by numerical uncertainties in the values of the parameters appearing in Eq. (26).

Let T be the thickness of the sand-layer, and let ϕ be the angle made with the horizontal by the plane of the layer. If T is not large, the temperature may be considered as constant across its width; then only the components of the temperature-gradient and of the acceleration of gravity *along* the inclined length enter the computations. Thus β' and g are reduced each by the factor $\sin\phi$. Let H be the inclined length of the layer. It should be noted in this connection that thickness is measured normally to the temperature-gradient, whereas in the analysis (Sec. I and II) this dimension was taken to be parallel to the temperature-gradient; this involves no change of formulae, for H merely corresponds directly with D. With these modifications of notation, Eq. (26) should be applicable.

As a numerical example, consider the Woodbine sand-formation of East Texas. Plummer and Sargent[7] give the following data for the region west of the Mexia Fault zone. In this area the Woodbine sand is 250 feet thick; and it has a uniform dip of 75 feet per mile, so that $\sin\phi = 0.0142$; it is abruptly terminated at its lower end by a fault-plane. The reciprocal temperature-gradient in this region is 50 feet per degree fahrenheit, so that $\beta'_{vert} = 3.64\times10^{-4}$ degrees centigrade per cm. H varies between 34 and 48 miles; and its mean value will be used here as 5.4×10^6 cm. Taking $\mu=1$ centipoise, $\rho_0=1$ g/cm³, $D=H$, $\alpha=8.5\times10^{-6}$ per degree C, and $h^2=0.009$ for sandy soil,[8] Eq. (26) specifies as the minimum value of the permeability for which convection currents may develop

$$k_{min} = 20 \text{ darcys}.$$

Muskat quotes several values for the permeability of the Woodbine sand, these values being in the neighborhood of 1 darcy. The implication of the present simple analysis and of the data listed above, therefore, is that convection currents should not occur in this portion of the Woodbine sand.

This conclusion concerning the existence of convection currents in the Woodbine sand west of the Mexia Fault zone, based as it is upon the rather limited accuracy of Eq. (26) (see the first paragraph of Sec. III), appears to be at variance with other evidence bearing upon the problem. One of the authors[9] has examined the distribution of NaCl in this region of the Woodbine sand. From geological considerations the

[7] F. B. Plummer and E. C. Sargent, Univ. Texas Bull., No. 3138 (1931).

[8] F. Birch, Editor, Geol. Soc. Am. Spec. Paper No. 36.

[9] C. W. Horton, Am. Assoc. Pet. Geol. Bull., 28, No. 11, p. 1635 (1944).

assumption that the source of NaCl is seepage along the fault-plane mentioned above is very probable. If this assumption be true, the distribution of NaCl can be explained by diffusion provided an "effective coefficient of diffusion" of 150 cm^2/day be admitted. The great excess of this 150 cm^2/day over the normal value of $\frac{1}{2}$ to 1 cm^2/day would seem, therefore, to argue for the existence of some NaCl-transport mechanism other than diffusion, a mechanism which is presumably convection currents.

If this interpretation of the NaCl distribution is correct, it may be tentatively concluded (a) that convection currents exist in the Woodbine sand west of the Mexia Fault zóne, and (b) that Eq. (26) predicts excessive temperature-gradients for the existence of convection currents when applied to subterranean sand-layers.

Pigment Dispersion Methods for Electron Microscopy

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A method of preparing a pigment dispersion for electron microscopy is described, wherein the pigment, such as zinc oxide, is first wet with water and is then dispersed in a solution of cellulose acetate. Another method is described in which mechanical or chemical damage of the particles is avoided by the use of an electrostatic dispersion apparatus.

A NUMBER of various types of pigment dispersion methods are now in use for electron microscopy. Certain fine powdered materials are successfully dispersed in water from which they may be settled out upon the surface of the conventional collodion membranes of specimen screens by drying. In the majority of instances, however, the surface tension effects upon the drying of water suspensions tend to agglomerate the material in clumps as the concentration increases by evaporation. Several wetting and dispersing agents have been employed to combat this effect, but none is very satisfactory. Almost all of these materials contain some non-volatile constituents which, in sufficient amounts, may remain to mar the particle outlines and haze the film.

The effect of agglomeration of pigment particles on drying of the suspension is not limited to the use of water, but also occurs with most non-aqueous materials such as the commonly used 2 percent collodion in amyl acetate. Fuller, Brubaker, and Berger[1] have effectively removed this difficulty by using, as described in their pigment mounting technique, the solvent, isopropyl acetate.

Suitable cellulose acetate solvents having the proper characteristics are scarce. It would appear that one important property of isopropyl acetate is that it has a many-fold water tolerance over amyl acetate. It has been found that 2-nitropropane in combination with a small percentage of acetone which has infinite water tolerance may be used with some success. Further we have found that a continuous film of cellulose acetate may be formed including particles of pigment which are actually dispersed in water. The advantages of such a technique are apparent in considering the lyophilic nature of most pigments. In this technique .01 gram of pigment is placed upon a clean glass plate. Just sufficient

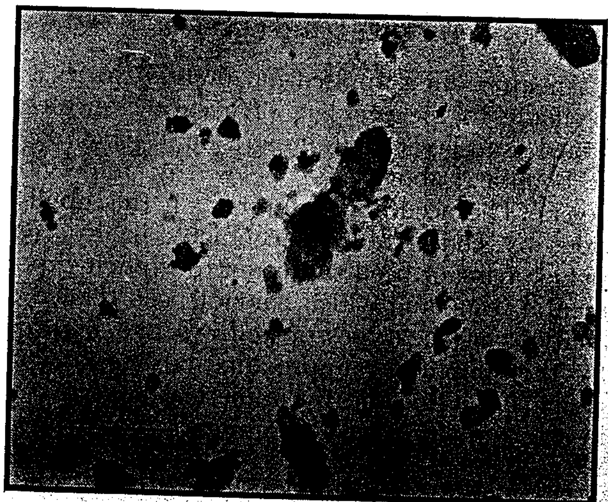

Fig. 1. Film including clay dispersion, methyl ethyl ketone method.

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[1] M. C. Fuller, D. G. Brubaker and R. W. Berger, J. App. Phys. 15, 201 (1944).