Advection-condensation of water vapor with coherent stirring: a stochastic approach

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Condensation of water vapour

- specific humidity of an air parcel:

\[ q = \frac{\text{mass of water vapor}}{\text{total air mass}} \]

- saturation specific humidity, \( q_s(T) \)
  - when \( q > q_s \), condensation occurs
  - excessive moisture precipitates out, \( q \to q_s \)
  - \( q_s(T) \) decreases with temperature \( T \)
  - \( T \) decreases with latitude \( \Rightarrow q_s \) position dependent

\[ q_s(T_1) \quad \text{and} \quad T_2 < T_1 \quad q_s(T_2) \]
Atmospheric moisture and climate

- Earth’s radiation budget:
  - absorption of incoming shortwave radiation generates heat
  - heat carried away by outgoing longwave radiation (OLR)
- water vapor is a greenhouse gas that traps OLR

- $\text{OLR} \sim -\langle \log q \rangle$

- $\text{OLR} \sim -\langle \log [\langle q \rangle + q'] \rangle \approx -\log \langle q \rangle + \frac{1}{2\langle q \rangle^2} \langle q'^2 \rangle$

- how fluctuation $q'$ is generated?
- what is the probability distribution of water vapor in the atmosphere?
Advection-condensation paradigm

Large-scale advection + condensation → reproduce (leading-order) observed humidity distribution

Observation

Simulation

- velocity and $q_s$ field from observation
- trace parcel trajectories backward to the lower boundary layer (source)
- track condensation along the way

ignore: cloud-scale microphysics, molecular diffusion, ...

(Pierrehumbert & Roca, GRL, 1998)
Advection-condensation model

- **PDE formulation:**
  \[
  \frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = S - C
  \]
  
  \(q\) is treated as a passive scalar advected by a prescribed \(\vec{u}\)

- **Particle formulation:**
  \[
  d\vec{X}(t) = \vec{u} \, dt, \quad dQ(t) = (S - C) \, dt
  \]
  
  Air parcel at location \(\vec{X}\) carrying specific humidity \(Q\)

- **\(S\) = moisture source (evaporation)**

- **\(C\) = condensation sink, in the rapid condensation limit**
  \[
  C : q(\vec{x}, t) \mapsto \min \left[ q(\vec{x}, t), q_s(\vec{x}) \right]
  \]

- **Saturation profile:**
  \[
  q_s(y) = q_0 \exp(-\alpha y)
  \]

- **\(y\) = latitude (advection on a midlatitude isentropic surface) or altitude (vertical convection in troposphere)**
**Previous analytical results**

1D stochastic models: $u \sim$ spatially uncorrelated random process

- **Pierrehumbert, Brogniez & Roca 2007**: white noise, $S = 0$

- **O’Gorman & Schneider 2006**: Ornstein-Uhlenbeck process, $S = 0$

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**Sukhatme & Young 2011**: white noise with a boundary source
Coherent circulation in the atmosphere

- moist, warm air rises near the equator
- poleward transport in the upper troposphere
- subsidence in the subtropics ($\sim 30^\circ$N and $30^\circ$S)
- transport towards the equator in the lower troposphere

Q: response of rainfall patterns to changes in the Hadley cells?
Steady-state problem

- bounded domain: \([0, \pi] \times [0, \pi]\), reflective B.C.
- \(q_s(y) = q_{\text{max}} \exp(-\alpha y)\): \(q_s(0) = q_{\text{max}}\) and \(q_s(\pi) = q_{\text{min}}\)
- resetting source: \(Q = q_{\text{max}}\) if particle hits \(y = 0\)

Cellular flow: \(\psi = -U \sin(x) \sin(y)\); \((u, v) = (-\psi_y, \psi_x)\)
Stochastic system with source

\[
\begin{align*}
\text{d}X(t) &= u(X, Y) \text{d}t + \sqrt{2\kappa} \text{d}W_1(t) \\
\text{d}Y(t) &= v(X, Y) \text{d}t + \sqrt{2\kappa} \text{d}W_2(t) \\
\text{d}Q(t) &= \left[ S(Y) - C(Q, Y) \right] \text{d}t
\end{align*}
\]

\[
\psi = -U \sin x \sin y \\
u = -\psi_y \\
v = \psi_x
\]

\[
U = 1 \\
\kappa = 10^{-2}
\]
**Bimodal distribution**: layer consists mainly of either:

- \( Q = q_{\text{min}} \) from upstream of the flow and diffuse in from the domain interior
- \( Q \approx q_{\text{max}} \) from the resetting source

particles with \( Q \approx q_{\text{max}} \) spreading into the domain as \( x \) increases
moist particles move up into region of low $q_s(y)$

at some fixed height $y_1$: mainly consists of $Q = q_{\text{min}}$ (diffuse in from the interior) and $Q = q_s(y_1)$ — Bimodal distribution

condensation $\Rightarrow$ localized rainfall over a narrow $O(\epsilon^{1/2})$ region
a homogeneous region of very dry air $Q \approx q_{\min}$ is created in the domain interior

the vortex "shields" the source from the interior

interior effectively undergoing stochastic drying
Steady-state Fokker-Planck equation for $P(x, y, q)$:

$$\epsilon^{-1} \vec{u} \cdot \nabla P - \partial_q [(S - C)P] = \nabla^2 P, \quad \epsilon = \kappa/(UL) \ll 1$$

Rapid condensation limit:

$$P(x, y, q) \neq 0 \quad \left\{ \begin{array}{l}
C = 0 \\
q \in [q_{\text{min}}, q_s(y)]
\end{array} \right\} \text{ for } x, y \in [0, \pi] \text{ and } q \in [q_{\text{min}}, q_s(y)]$$

Resetting source at bottom boundary:

$$P(x, y = 0, q) = \pi^{-1} \delta(q - q_{\text{max}})$$

At the top boundary: $P(x, y = \pi, q) = \pi^{-1} \delta(q - q_{\text{min}})$

Hence,

$$\epsilon^{-1} \vec{u} \cdot \nabla P = \nabla^2 P$$

which predicts a boundary layer of thickness $O(\epsilon^{1/2})$
1. domain interior, to leading-order:

\[ P_0 = \pi^{-2} \delta(q - q_{\text{min}}) \]

2. source boundary layer:

\[ P_0 = G(x, y) \delta(q - q_{\text{min}}) + H(q, x, y) \]

3. condensation boundary layer:

\[ P_0 = G(x, y) \delta(q - q_{\text{min}}) + [\pi^{-2} - G(x, y)] \delta(q - q_{s}(y)) \]

In the \( O(\epsilon^{1/2}) \) boundary layers, introducing coordinates (Childress 1979):

\[ \zeta = \epsilon^{-1/2} \psi \quad \text{and} \quad \sigma = \int |\nabla \psi| \, dl, \quad l = \text{arclength} \]

Equation for \( G(\sigma, \zeta) \) reduces to:

\[ \partial_{\sigma} G = \partial_{\zeta} \zeta G \]
Mean moisture input rate $\Phi$

\[ \Phi = \varepsilon^{-1/2} \sqrt{\frac{8\kappa}{\pi}} (q_{\text{max}} - q_{\text{min}}), \quad \varepsilon = \kappa/(UL) \]

Other diagnostics: horizontal rainfall profile, vertical moisture flux, … etc