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How is a Collection Related to its Members?

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Fundamental Relations in Ontology

Much discussed:

- Parthood
- Instantiation (of a class by an individual)
- Subsumption (of a class by another class)
- Constitution (of one individual by another)
- Dependence (of one individual/class on another)

Comparatively neglected:

- Membership (of an individual in a collection)

Example: A Choir

If you see a choir, then you see some singers; and if you see those singers, you see a choir.

What is the relationship between the choir and the singers?

Obvious answer: The Choir = The Singers.

Problem: A singular entity (the choir) cannot be identical to some plural entities (the singers):

$$\text{The Choir} \neq \text{The Singers.}$$

Making the plural singular?

If we are to relate the choir to the singers using a statement of the form $X = Y$, then the (singular) choir must be equated to some singular entity that is in some way dependent on the singers:

$$\text{The Choir} = \text{F(The Singers)}$$

where F is a way of specifying a singular entity in terms of some plurality.

[Note how hard it is to *say* this: "The plurality" already seems to denote something singular — the collection rather than its members.]

What can F be?

First Candidate: Sets

Can the collection be equated to the *set* of its members, in the mathematical sense?

Two good reasons why not:

- Sets are abstract, collections are concrete. The choir has a physical location, can be seen, emits sounds. No mathematical set can do any of these things.
- Sets have fixed membership, collections can lose or gain members.

Note: You can't "add a member to a set". What you can do is consider a *different* set whose members are those of the original set plus one extra.

Hence,

$$\text{The Choir} \neq \{x : x \text{ is one of the singers}\}$$

Two senses of 'membership'

Membership in a collection is not the same as membership in a set.

The former has priority since it accords with ordinary language (e.g., members of a choir, a club, a political party). Better to talk about *elements* of a set.

For set elementhood we use the usual notation $x \in X$.
For membership in a collection, we use $Member(x, y)$.

Then for a collection c, we can put

$$c \neq \{x : Member(x, c)\}.$$

The relation of x to these two things is different:

$$Member(m, c) \text{ but } m \in \{x : Member(x, c)\}.$$

The time factor

Since membership in a collection can vary over time, the truth-value of '$Member(x, y)$' will vary.

An obvious way to handle this is to show the temporal dependency explicitly as

$$Member(x, y, t).$$

For each time at which the choir exists, we can form the set of individuals who are members of the choir at that time:

$$\forall t(Exists(c, t) \rightarrow \exists S(Set(S) \wedge \forall y(Member(y, c, t) \leftrightarrow y \in S))).$$

We cannot put $c = S$ here, since this would imply

$$Member(y, c, t) \leftrightarrow Member(y, c, t'),$$

contradicting the time-dependence of membership.

Second candidate: Sums

The mereological sum (*fusion*) of a set of objects S is that object σS which overlaps all and only those objects which overlap some element of S:

$$\forall x(\mathit{Overlaps}(x, \sigma S) \leftrightarrow \exists y(y \in S \wedge \mathit{Overlaps}(x, y)))$$

where

$$\mathit{Overlaps}(x, y) =_{\mathrm{def}} \exists z(\mathit{PartOf}(x, z) \wedge \mathit{PartOf}(z, y)).$$

Can we identify the choir with the sum of its members?

Is the choir a sum of singers?

- Unlike a set of singers members, the sum of the members of the choir is a concrete physical entity, located in space, like the choir itself.
- But like a set of singers, a sum of singers is temporally unvarying. It is *mereologically constant* — if new members join the choir, we have the same choir but a different sum of members.

At each time the choir exists, there is an object that is the sum of its members at that time:

$$\forall t(\mathit{Exists}(c) \rightarrow \exists s(s = \sigma\{x : \mathit{Member}(x, c, t)\})),$$

Can we put $c = s$ here?

The choir is not a sum of singers

If we put $c = s$, i.e.,

$$c = \sigma\{x : Member(x, c, t)\}$$

then to avoid singling out t preferentially we must also put

$$c = \sigma\{x : Member(x, c, t')\}$$

for any other time t' at which the c exists.

We then have

$$\sigma\{x : Member(x, c, t)\} = \sigma\{x : Member(x, c, t')\}$$

in contradiction to the variable membership of c.

Hence $c \neq s$, i.e.,

The Choir $\neq$ The sum of members of the Choir

Summary so far

We have four entities, as follows:

The choir
The singers
The set of singers
The sum of the singers

No two of these are equal; they differ with respect to one or more of numerosity, physicality, and mereological constancy:

- The choir is singular, physical, and mereologically variable.
- The singers are plural, physical, and mereologically variable.
- The set of singers is singular, abstract, and mereologically constant.
- The sum of singers is singular, physical, and mereologically constant.

Why are we looking for an identity?

We are trying to characterise the relationship between a collection and its members.

We have tacitly assumed that this can be elucidated by exhibiting an identity between the collection and some singular entity derived in some way from the members.

Of course we can do this: the collection is identical with the collection of its members. But this begs the question unless "collection of" is further elucidated.

If we cannot equate the choir with any singular entity dependent on its members, then perhaps we can find a suitable relation that falls short of identity. There are many candidates in the literature.

Candidate solution I: Eliminativism

In its starkest form, eliminativism says that *there are no compound entities*. Statements about compounds are to be paraphrased into statements about simples configured and interacting in certain ways.

There are no tables, chairs, people, choirs, . . . , just simples tabling, chairing, peopling, choiring . . . . (But Wittgenstein in the *Tractatus* was notoriously reluctant to say what the simples were.)

More moderate forms of 'selective' eliminativism are possible. (E.g., Van Inwagen is eliminativist with respect to non-living things, but accepts the existence of organisms.)

Can we be eliminativist towards collections while accepting the existence of objects like tables, chairs, and people?

Collection-eliminativism

On this view, there are the Singers but not the Choir.

"The choir is singing" means "The singers are singing chorally".
"The choir was disbanded" means "The singers ceased to relate to one another chorally".
Etc, etc.

Our problem — the relationship between the collection and its members — goes away . . .

. . . but only at the cost of having to specify how everything we can say about a collection can be paraphrased without reference to collections.

We also would need to specify where the line is drawn between collections and compound objects.

Candidate solution II: Constitution

The choir is *constituted* by the sum of the singers.

(Likewise, a bicycle is constituted by the sum of its components, the vase is constituted by a quantity of clay.)

Constitution is not identity, so

- the same choir can be constituted by different member-sums at different times.
- the same member-sum may constitute different choirs (or other collections) at different times (or not constitute anything at all).

The relationship between the choir and the sum of its members is *unity without identity* (Lynne Rudder Baker).

Constitution is "multiplicativist"

Constitution is in many ways an attractive solution, but has been criticised as *multiplicative*:

When the choir is on the stage, a separate thing, a sum of singers, is also there, occupying exactly the same place. Yet we only seem to see one thing in that place.

The stance toward constitution is one of the major differences between the BFO and DOLCE ontologies: DOLCE embraces constitution whereas BFO rejects it.

Candidate solution III: Four-dimensionalism

There are two flavours of four-dimensionalism:

1.
   - The objects we refer to using nouns like 'table', 'person', and 'choir' are extended in time as well as space. They are four-dimensional *hyperobjects*.
   - At any one time, what is present is not the whole choir but a thin cross-section of it, perpendicular to the time axis.
   - At different times I see the same choir by seeing different cross-sections of it.
2.
   - 'Choir', 'table', 'person' refer to *stages* of histories, where a history is the complete content of some spatio-temporal region.
   - Different stages of the same choir-history are distinct choirs.
   - Thus, as with Heraclitus' river, I never see the same choir twice.

These are distinct four-dimensionalist accounts, but both can solve the problem in the same way.

The Choir as Hyperobject

The relation between the choir and the sum of singers is *spatio-temporal overlap*: in particular, they share at least one cross-section.

Identity between hyperobjects is straightforward: $x_1 = x_2$ if and only if they have the same spatio-temporal extension.

For hyperobject x, write x^t to denote its cross-section at time t.

Then for the choir c and the sum-of-singers σS, at times t_1 and t_2, we can consistently put

$$\begin{aligned} c^{t_1} &= (\sigma S)^{t_1} \\ c^{t_2} &= (\sigma S)^{t_2} \\ c^{t_1} &\neq c^{t_2} \\ (\sigma S)^{t_1} &\neq (\sigma S)^{t_2} \end{aligned}$$

Is Four-dimensionalism Coherent?

Probably yes.

But it comes at a cost: one must radically reinterpret large parts of our everyday language. (Specifically, anything to do with change.)

Candidate solution IV: Temporal Identity

We distinguish *synchronic identity* ($a \stackrel{t}{=} b$) from *diachronic identity* ($a = b$).

This has not been popular with philosophers. Doubt has been cast on its coherence. But let's see if we can make anything of it.

The idea is that, in the words of Gallois, there can be *occasions of identity*. That is: a and b might be (synchronically) identical on one occasion without being (diachronically) identical on all occasions.

The choir is synchronically identical to one sum of singers at t_1 and to another sum of singers at t_2.

Let us investigate ...

Properties of synchronic identity

Being an identity relation, synchronic identity must be an equivalence relation — except that reflexivity is restricted to times at which the object exists:

TEQR: $Exists(x, t) \rightarrow x \stackrel{t}{=} x$

TEQS: $x \stackrel{t}{=} y \rightarrow y \stackrel{t}{=} x$

TEQT: $x \stackrel{t}{=} y \wedge y \stackrel{t}{=} z \rightarrow x \stackrel{t}{=} z$

(If $Exists(x, t)$ is *defined* as $x \stackrel{t}{=} x$ then TEQR can be dropped.)

Relation between synchronic and diachronic identity: Anything existing at t is synchronically identical at t to anything that it is diachronically identical to:

$$\forall t(Exists(x, t) \rightarrow (x = y \rightarrow x \stackrel{t}{=} y)).$$

Leibniz's law

For diachronic identity, Leibniz's law is the usual

$$x = y \rightarrow \forall F(F(x) \leftrightarrow F(y)).$$

For synchronic identity, F must be restricted to *synchronic properties*, i.e., those properties 'the instantiation of which at t does not entail the instantiation of any property at any other time' (Doepke):

$$x \stackrel{t}{=} y \rightarrow \forall F \in \mathcal{SP}(F(x,t) \leftrightarrow F(y,t)),$$

where $\mathcal{SP}$ is the set of predicates $F(x,t)$ expressing synchronic properties.

Synchronic identity and collections

At each time of its existence, a collection is synchronically identical to the sum of its members at that time.

Let c be the choir, and suppose it has different members at t_1 and t_2. Then we can consistently put

- $c \overset{t_1}{=} \sigma\{x : Member(x, c, t_1)\}$
- $c \overset{t_2}{=} \sigma\{x : Member(x, c, t_2)\}$
- $\sigma\{x : Member(x, c, t_1)\} \neq \sigma\{x : Member(x, c, t_2)\}$

Since sums are mereologically constant, the third formula implies

- $\forall t \neg(\sigma\{x : Member(x, c, t_1)\} \overset{t}{=} \sigma\{x : Member(x, c, t_2)\})$

Members and parts of collections

If a collection exists at t, there must be a sum which it is synchronically identical to at t:

COL $\quad Col(x) \wedge Exists(x, t) \rightarrow \exists S(Set(S) \wedge x \stackrel{t}{=} \sigma S)$

(where $Col(x)$ is read 'x is a collection).

However, from σS it is not possible to retrieve the elements of S, since different sets can have the same sum.

Therefore to characterise the members of a collection we must specify what kind of thing it is a collection of.

What Collections are Collections Of

Write $ColOf(x, y)$ to mean that x is a collection of elements of class y, i.e., each member of x is an instance of class y.

Axioms for *ColOf*

COLOF1: $ColOf(x, y) \rightarrow Col(x) \wedge Class(y)$

COLOF2: $Col(x) \rightarrow \exists y ColOf(x, y)$

COLOF3: $ColOf(x, y) \wedge IsA(y, z) \rightarrow ColOf(x, z)$

COLOF4: $ColOf(x, y) \wedge ColOf(x, z) \rightarrow \exists w(IsA(w, y) \wedge IsA(w, z) \wedge ColOf(x, w))$

COLOF5: $ColOf(x, y) \wedge Exists(x, t) \rightarrow \exists S(Set(S) \wedge x \stackrel{t}{=} \sigma S \wedge \forall z(z \in S \rightarrow InstanceOf(z, y)))$

Defining Membership of a Collection

If x is a collection of elements of type w then the members of x at time t are those parts of x which are of type w:

$$\mathit{Member}(y, x, t) \;=_{\text{def}}\; \mathit{PartOf}(y, x, t) \wedge \forall z(\mathit{ColOf}(x, z) \rightarrow \mathit{InstanceOf}(y, z)).$$

Note: This requires the *PartOf* relation to be relativised to times. The antisymmetry rule, if required, would then be

$$\mathit{PartOf}(x, y, t) \wedge \mathit{PartOf}(y, x, t) \rightarrow x \overset{t}{=} y$$

Is Temporal Identity Coherent?

Thus far, it seems to be, so long as one is prepared to countenance stronger and weaker forms of identity.

But arguably, identity is *unique*: there can only be one identity relation.

In which case, temporal identity must be replaced by something else, e.g., constitution (or, the symmetric closure of constitution), which brings us back to multiplicativism.

Final Unanswered Question

What is there to a collection over and above a *bare plurality*?

A plurality exists so long as some things exist which are its constituent individuals.

A "genuine" collection should have some conditions attached to membership — e.g., that the elements making up the collection are related in some way.

Such conditions can lend coherence to a collection so that it can be treated as an individual in its own right. For an informal discussion, see Wood and Galton, 'A Taxonomy of Collective Phenomena' (*Applied Ontology*, 2009). But the formal work remains to be done.

Conclusions

- A collection cannot be identified with its members, with the set of its members, or with the sum of its members.
- We considered various alternative ways of accounting for the relationship between a collection and its members:
  - Eliminativism (collections don't exist)
  - Constitution (a collection is constituted by the sum of its members)
  - Four-dimensionalism (collections and member-sums are partially overlapping four-dimensional entities)
  - Temporal identity (at each time a collection is synchronically identical to the sum of its members at that time) We explored the last of these in more detail as it is a comparatively neglected possibility.
- Unfinished business: How to distinguish a collection from a "bare plurality"