How is a Collection Related to its Members?

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Abstract. It is increasingly recognised that a comprehensive upper ontology should provide an account of collections as well as individual entities. A prerequisite for this is to characterise the relationship between a collection and its members. In this paper various possible accounts are discussed, including eliminativism, constitution, and four-dimensionalism. In particular, the idea of making a distinction between synchronic and diachronic identity relations, which has mostly been regarded with disfavour by philosophers and ontologists, is explored.

Foundational or 'upper' ontology concerns itself with those categories and relations that enjoy a sufficient degree of universality to be deployable across all or most of the specific subject domains for which an ontology may be required. The key ontological relations include **parthood** ($PartOf$), **instantiation** of a class by an individual ($InstanceOf$), and **subsumption** of one class by another (IsA), and these are recognised by all proposals for a foundational ontology. Other fundamental relations which are very frequently encountered are **constitution** (of some 'higher-level' entity by a 'lower-level' entity) and **dependence**. Comparatively neglected is the relationship of **membership** between an object and a collection to which it belongs; yet it is increasingly recognised that any reasonably comprehensive upper ontology should provide an account of collections. In this paper, I will discuss various proposals for characterising the relationship of a collection to its members.

To focus our ideas, let us consider the example of a choir, which may be regarded as a collection of singers — not an arbitrary collection of singers, of course, but a group of singers who have formed an agreement to come together for the purpose of performing certain types of musical works. Imagine you are at a performance by this choir: you see them on the stage, singing. You also see singers on the stage. The question is: what is the relationship between the choir that you see and the singers that you see?

An obvious first response is: the singers *are* the choir. As far as it goes this seems to be acceptable, but if we wish to establish our ontological ideas on a rigorous formal footing we have to look below the surface and, specifically, ask what exactly we mean by 'are' here. If the singers are the choir, then presumably also the choir is the singers, and the difference in number between 'is' and 'are' already points to a problem. The choir is singular and the singers are plural, and therefore we cannot simply equate them. Thus the 'is/are' by which we relate the choir and the singers cannot be simple identity. Thus we can put

$$\text{the choir} \neq \text{the singers}.$$

If the choir is to be equated in some way with the singers, using a statement of the form $X = Y$, then what the choir is thereby equated to cannot be the plural singers but some *singular* entity which is in some way dependent on the singers. We must therefore consider what ways there are of specifying a singular entity that is dependent on a given plurality. Two possibilities spring immediately to mind, namely *sets* and *sums*. Let us consider these in turn.

We can fairly quickly discount sets as the kinds of singular entity we are looking for. The set of singers (specifically, the set of all and only those singers present on that particular stage at that particular time) is a mathematical abstraction to which each of the individual singers stands in an abstract relation called *membership*, designated $\in$. This use of the term 'member' is unfortunate, since the same term is already used in many everyday contexts (e.g., the members of a family, a club, a university — or, indeed, a choir), and it would be begging a great many questions if we were to assume that the relation or relations designated by the term in these everyday contexts were the same as the relation designated in mathematical set theory by the symbol '$\in$'. For this reason, I shall henceforth always speak of the *elements* of a set rather than its members.

There are two reasons why we cannot identify the choir with the set of its members. One is simply that the set is, as already noted, abstract: a set does not have a physical location, cannot be seen, and emits no sounds, whereas a choir does all of these things. Here I must emphasise that I am talking about mathematical sets; the word 'set' is also used — and this is surely its primary usage in English — to refer to a physical collection, e.g., a chess set, a tea set, or a set of spanners. None of these is a mathematical set, although for each of them one can form the mathematical set of its constituent entities.

The second reason why we cannot identify the choir with the set of its members is that the members of a choir can change but the elements of a set cannot. You cannot add a new element to a set; the nearest thing to this that you can do is to consider a *different* set whose elements are all the elements of the original set together with the new element. The second set is not the same set as the first. But when a new member joins a choir, it is the same choir before and after.

For these reasons, then, we can assert that

$$\text{the choir} \neq \{x : x \text{ is one of the singers}\},$$

or, to give it in a more readily generalisable form,

$$\text{the choir} \neq \{x : x \text{ is a member of the choir}\}.$$

What we *can* say is that for each time at which the choir exists, there is a set which is the set of its members at that time:[1]

$$\forall t(Exists(choir, t) \rightarrow \exists S(Set(S) \wedge \forall y(Member(y, choir, t) \leftrightarrow y \in S))),$$

but we cannot, in this formula, add a conjunct $choir = S$. Notice, crucially, that the predicate $Member$ used in this formula has a temporal argument, reflecting

[1] The condition $Exists(choir, t)$ could be dropped, since for the times when the choir does not exist, we can put $S = \emptyset$ (as noted in [1]).

the fact that membership of a collection can change. This is in contrast to the mathematical relation $\in$, which is not time-dependent and therefore does not have a temporal argument.

Our second candidate for identification with the choir is the *sum of its members*. By sum I mean the mereological sum (or fusion). According to the standard definition, the mereological sum of a set of objects S is that object σS which overlaps all and only those objects which overlap some element of S; thus we have

$$\forall x(Overlaps(x, \sigma S) \equiv \exists y(y \in S \wedge Overlaps(x, y))),$$

where the relation *Overlaps* is, as usual, defined by

$$Overlaps(x, y) =_{\text{def}} \exists z(PartOf(z, x) \wedge PartOf(z, y)).$$

Of the two objections to identifying the choir with the *set* of its members, the first would not apply to identifying the choir with the *sum* of its members. For unlike the set, the sum is a concrete, physical entity: the sum of a set of singers has a location, can be seen, and can emit sounds — just like the choir, in fact, which lends some plausibility to the identification of the two. Unfortunately, however, the second objection to the set idea applies equally to the sum idea: just as the elements of a set are unchanging, so the sum is *mereologically constant*. Thus if, after I have seen the choir on a particular occasion, some members leave, and new members join, then the next time I see them I will be looking at a different sum; and the sum I saw before will no longer be a choir.

Actually, we have to be a little careful here: whereas a set is always a set *of* particular individuals (its elements), a sum is just a sum. A sum of singers is also a sum of atoms, by virtue of the fact that each singer is made of atoms: there is no privileged decomposition of the sum into singers, atoms, or any other kinds of unit. This is in stark contrast to the set case, where the set of singers in the choir is actually disjoint from the set of atoms in the choir. When I look at the choir on one occasion, I see a sum which is simultaneously the sum of a set of singers and the sum of a set of atoms. If I see the same choir a week later, then even if there has been no change of membership of the choir (so the set of singers in the choir now is the same as the set of singers in the choir earlier) I will not be looking at the same sum since over the intervening week each singer will have gained some atoms and lost others.

Just as in the set case, we can say that at each time the choir exists, there is an object which is the sum of its members at that time:[2]

$$\forall t(Exists(choir, t) \rightarrow \exists s(s = \sigma\{x : Member(x, choir, t)\})),$$

but again, we cannot add a conjunct $choir = s$. In other words, we have

$$\text{the choir} \neq \sigma\{x : x \text{ is a member of the choir}\}.$$

[2] Note that in this case we cannot drop the condition $Exists(choir, t)$ — at least not on the commonly accepted assumption that mereology should not admit a null element.

It should be noted, incidentally, that there is a way of understanding 'the sum of its members' which overrides the temporal objection to identifying the choir with the sum of its members. If the choir is the sum of its members, then for any property P, the choir has P if and only if the sum of its members has P. Thus if the choir came into existence in 1993, then the sum of the members of the choir must have come into existence in 1993 too. If we give 'the sum of the members of the choir' a *de dicto* reading then this is trivially true: before 1993 there was no sum of the members of the choir, because there was no choir then and therefore no members. But this was not what we meant: we were interested in the possibility that the choir could be identified with a particular quantity of matter, and this identification fails because the history of the choir is distinct from that of any quantity of matter.[3]

To sum up what we have found so far: A collection cannot be identified with its members, the set of its members, or the sum of its members. What is left? It appears that the collection is *sui generis*, and cannot be identified with anything that we can specify independently.

Instead of seeking an identity, perhaps we should rather be looking for some other relationship which a collection can bear to some entity specifiable in terms of the collection's members. For this purpose it is worth examining two problems which are in many ways analogous to ours, and which have been more thoroughly discussed in the literature. For the relationship between a collection and its members is at least analogous to the relationship between a material object and the matter that it is made of, and the relationship between an assembly (i.e., an artefact produced by putting together various components in a structured way) and its components. In relation to these cases, there is an extensive philosophical literature, in which, indeed, the case of collections has often been mentioned, although they have seldom been the main focus of the discussion.

It is not my intention to summarise this literature here — I have neither the time nor the competence to do so. Instead I shall discuss just a few of the major proposals which have been made, and examine specifically how they fare in relation to my primary topic of collections.

In the first place, it is widely (though by no means universally) agreed that a material object cannot simply be identified with the matter it is made of, essentially because the object and the matter can have different histories, as famously illustrated by the statue and the clay, where the clay existed before, and will outlive, the statue. Similarly, an assembly cannot be identified with the sum of its components, for a similar reason — the most famous example in this case being the Ship of Theseus. Similar considerations led us to the conclusion

[3] There is an assumption here that it makes sense to speak of the matter independently of anything that it is the matter of. For Fine [2], however, the fusion of a set may be either an *aggregate*, which exists for as long as at least one element of the set exists, or a *compound*, which exists only when *all* the elements of the set exist; but not, apparently, a sum in the sense intended here, which picks out the matter of which elements of the set are composed, and may persist without reference to the continuing existence or otherwise of the elements themselves.

that that the choir cannot be identified with the sum (or set — but for the time being we will assume that the entity in question is the sum) of its members.

Let us now consider what relations, falling short of identity, have been proposed to handle these cases. We shall consider in turn: eliminativism, the constitution view, four-dimensionalism, and temporal identity.

The *eliminativist* solution to the problem is on the face of it straightforward: compound objects simply do not exist.[4] There are no statues, ships, choirs, or singers. All that exists are simples, and statements about statues (etc.) must all be paraphrased as statements about simples statuizing (etc.), where for simples to statuize is for them to configure themselves and behave in a particular way which we, in our ontological confusion, describe as there being a statue (etc.). This idea applies equally well to everyday objects, assemblies, and collections, and in some ways one might find it more plausible as an account of collections than of the others, since the members of a collection are by nature less cohesive than the components of an assembly or the matter of a material object. Even so, it is not clear what is achieved by this, even supposing all the necessary paraphrases are carried out (and it is far from clear that this will actually be possible). As well as turning common nouns into verbs (e.g., 'choir' into 'choirize') one would have to turn proper nouns into adverbs (thus 'Exeter Festival Chorus' becomes 'choirizing Exeter-Festival-ly', where these neologisms have to be spelt out without referring to Exeter, festivals, or choirs). As a matter of fact, I am not convinced that there is any knock-down argument against eliminativism; but I see it as somewhat like solipsism. Just as Wittgenstein pointed out that

> ... solipsism, when its implications are followed out strictly, coincides with pure realism. The self of solipsism shrinks to a point without extension, and there remains the reality coordinated with it.
>
> *Tractatus Logico-Philosophicus, 5.64*

so, within eliminativism, we have to 'rebuild' all our usual concepts, with their associated ontological problems, through the paraphrases offered for the terms we ordinarily use to describe them.

Under the *constitution* view [3, 5, 4], we say that the choir is constituted by the sum of its members, the bicycle is constituted by the sum of its components, and the vase is constituted by the clay. Since constitution is not identity, the choir can be constituted by different member-sums at different times, the bicycle by different component-sums, and even, the vase by different quantities of clay (e.g., if a piece was broken off and repaired using fresh clay). And contrariwise, the members whose sum now constitutes this choir may on another occasion (or even simultaneously) constitute an orchestra or the music department of a university; and the portion of matter which now constitutes the vase previously constituted a formless lump, and later will constitute a heap of fragments.

[4] Doepke [3] refers to this as *reductivism*, but Baker [4] reserves the latter term for the doctrine that statues, ships, etc, *do* exist but are nothing but aggregates of simples.

A frequent objection to the constitution view is that it is *multiplicativist*. If the choir is not the sum of its members, then when the choir is present on stage, say, there are then two distinct things on stage, namely the choir and the singers. In the terminology of [4], the relationship between the choir and the group of singers is *unity without identity*. One wants to say that, on the contrary, the choir just *is* that particular group of people on the stage. On an earlier occasion, the same choir *was* a different group of people. The problem is that if the choir is both the first group and the second group, then, if the identities here are strict (hence transitive), the first group must be the second group. Although such considerations may seem to be abstrusely philosophical, they play a formative role in the development of formal ontologies for application to information systems. Some ontologies, such as DOLCE, explicitly adopt a multiplicativist approach; others, such as BFO, explicitly repudiate it [6].

For those who would repudiate multiplicativism, one response to this is to adopt *four-dimensionalism* [7, 8]. Four-dimensionalists hold that the standard notion of a continuant is incoherent. The entities we refer to using nouns and definite descriptions are actually extended in four dimensions.[5] What we see at any one moment is an extremely thin cross-section of the entity, at right-angles to the time axis (and the theory of relativity tells us that observers in different states of motion will cut these sections somewhat skew to one another — but that is another story, which I do not wish to go into here). Four-dimensional objects (let us call them *hyperobjects*) have a robust criterion of identity: they are equal if and only if they coincide over their entire four-dimensional extent. Then the relation between a particular group of people and the choir that for a period of time they form is simply one of *overlapping*: that is, certain cross-sections of the 'choir' hyperobject (c) coincide with the contemporaneous cross-sections of the 'sum of people' (σS) hyperobject. If each cross-section of a hyperobject is itself regarded as an object, then the cross-section of c at t_1, say, is identical to the cross-section of σS at t_1, but the cross-section of c at t_2 is not identical to the cross-section of σS at t_2. There is no contradiction: writing X^t for the cross-section of X at t, we have

$$c^{t_1} = (\sigma S)^{t_1}, c^{t_2} \neq (\sigma S)^{t_2}, c^{t_1} \neq c^{t_2}, (\sigma S)^{t_1} \neq (\sigma S)^{t_2}.$$

This picture seems to be perfectly coherent, but it comes at a high cost: one can only adopt four-dimensionalism if one is prepared to radically reinterpret almost all our everyday language — most obviously, to understand by *change* in an entity, *difference* between distinct temporal cross-sections of the entity.

What other options exist for the non-multiplicativist who is determined at all costs to avoid invoking constitution? Here I want to examine — if only because it is generally frowned upon — the possibility of distinguishing *synchronic* and *diachronic* forms of identity. The idea is that identity at a time is distinct from identity over time. Can anything coherent be made of this?

[5] Or, alternatively, they are *stages* of such four-dimensional entities, where the understanding is that distinct stages of the same four-dimensional entity are non-identical. In the discussion here, I do not consider the stage theory.

Objects are now three-dimensional again: traditional continuants. I shall write $a \overset{t}{=} b$ to mean that at time t, a is identical to b; this relation has been called Temporal Identity [3] or Occasional Identity [9]. I want to use this where the multiplicativists would say that one of a and b constitutes the other. What are the rules for $\overset{t}{=}$?

Being a form of identity, we naturally require $\overset{t}{=}$ to be an equivalence relation: reflexive, symmetric, and transitive. However, we must be careful: if x does not exist at t, then we cannot say $x \overset{t}{=} x$. Hence we must qualify the reflexivity rule so that $\overset{t}{=}$ is an equivalence relation on the set of objects existing at t. Hence we have

TEQR: $Exists(x, t) \to x \overset{t}{=} x$
TEQS: $x \overset{t}{=} y \to y \overset{t}{=} x$
TEQT: $x \overset{t}{=} y \wedge y \overset{t}{=} z \to x \overset{t}{=} z$

We could indeed *define* $Exists(x, t)$ to be equivalent to $x \overset{t}{=} x$ (call this definition DEQ), in which case TEQR need not be postulated separately.

How do synchronic and diachronic identity interact? The basic rule, which follows from DEQ, is that if x exists at t, then it is synchronically identical at t to anything that it is diachronically identical to:

$$x = y \to \forall t(Exists(x, t) \to x \overset{t}{=} y).$$

Identity is characterised by Leibniz's law, which for diachronic identity is:

LLEQ: $x = y \to \forall F(F(x) \leftrightarrow F(y))$

For synchronic identity, we must restrict F to be, in the words of Doepke [3], "any property, the instantiation of which at t does not entail the instantiation of any property at any other time". Let us call these "synchronic properties" and write $\mathcal{SP}$ for the set of all predicates (of the form $F(x, t)$) which express synchronic properties. Assuming F is thus restricted, we put

LLTEQ: $x \overset{t}{=} y \to \forall F \in \mathcal{SP}(F(x, t) \leftrightarrow F(y, t))$.

Doepke ridicules this by suggesting that it opens up the possibility of defining other, bizarre, notions of identity by introducing arbitrary restrictions on the class of predicates for which the corresponding version of Leibniz's law is to hold. But this seems to me unconvincing: time is special, indeed unique, and there seems to me to be no justification, given a form of definition which accords time a special role, for generalising it to other definitions in which an analogous role is played by other, less central, concepts.

How does synchronic identity apply to collections? An attractive possibility is to say that at each time of its existence, a collection is synchronically identical to the mereological fusion of its members: more exactly, at each time of its existence, there is a set of objects ("its members") of which it is the fusion. Writing $Col(x)$ to mean that x is a collection, and σS for the fusion of the elements of set S, COL: $Col(x) \wedge Exists(x,t) \rightarrow \exists S(Set(S) \wedge x \stackrel{t}{=} \sigma S)$

As noted above, we cannot, from a sum σS, retrieve the elements of the set S, since in fusing the elements of S we lose track of their identities. For this reason, even though membership of a collection is surely a form of parthood (as acknowledged, for example, by [10]), we cannot simply say that the members of collection are its parts.

To characterise the members of a collection, we use the idea that any collection is a collection *of* elements of some specified class. I shall write $\mathit{ColOf}(x,y)$ to mean that x is a collection of ys, i.e., its members are all instances of class y. This relation is provisionally axiomatised as follows:

COLOF1: $\mathit{ColOf}(x,y) \rightarrow Col(x) \wedge Class(y)$
COLOF2: $Col(x) \rightarrow \exists y \mathit{ColOf}(x,y)$
COLOF3: $\mathit{ColOf}(x,y) \wedge IsA(y,z) \rightarrow \mathit{ColOf}(x,z)$
COLOF4: $\mathit{ColOf}(x,y) \wedge \mathit{ColOf}(x,z) \rightarrow \exists w(IsA(w,y) \wedge IsA(w,z) \wedge \mathit{ColOf}(x,w))$
COLOF5: $\mathit{ColOf}(x,y) \wedge Exists(x,t) \rightarrow \exists S(Set(S) \wedge x \stackrel{t}{=} \sigma S \wedge \forall z(z \in S \rightarrow \mathit{InstanceOf}(z,y)))$

Note that COL now follows from COLOF2 and COLOF5. COLOF3 says that if ys are zs then a collection of ys is a collection of zs — so e.g., a collection of dogs is a collection of animals. COLOF4 says that if a collection of ys is also a collection of zs, then it is a collection of elements of some common subclass w of y and z; thus a collection of dogs can be a collection of pets (by being a collection of pet dogs) but not a collection of atoms (since no dog is an atom). Of course, at a particular time, a collection of dogs will be synchronically identical to some sum of dogs, and therefore to a sum of atoms, and therefore to a collection of atoms, but since these identities are only synchronic we cannot equate the two collections using '=' .

We can now identify the members of the collection x at time t as those parts of x which are of the right type:

$$Member(y,x,t) =_{\text{def}} \mathit{PartOf}(y,x,t) \wedge \forall z(\mathit{ColOf}(x,z) \rightarrow \mathit{InstanceOf}(y,z)).$$

Note that this uses a temporally relativised form of parthood, such as is used by Simons in his system CT [11, §5.2]. Given our notion of synchronic identity, it seems natural to postulate a form of anti-symmetry, as follows:

TPA: $\mathit{PartOf}(x,y,t) \wedge \mathit{PartOf}(y,x,t) \rightarrow x \stackrel{t}{=} y.$

This goes beyond what Simons is prepared to countenance: on the right-hand side, he uses a notion of 'coincidence' which is not regarded as a form of identity (for example, it does not obey even our temporally restricted form of Leibniz's law) — it is more like the symmetric closure of the constitution relation.

Thus far, the notion of synchronic identity seems coherent, and can support a characterisation of the membership relation as it applies to collections. This is, of course, only a beginning, and the further exploration of this relation will need to consider what there is to a collection over and above a "bare plurality" (such as, for example, *the people born in 1952*), for which no condition is to be attached to its existence over and above the identification of its constituent individuals. A person born in 1952 automatically qualifies, by virtue of that fact alone, as a member of the bare plurality designated by *the people born in 1952*. A collection is also a plurality, but it is more than that. For a plurality to be a collection, its constituents must stand in some relation of association which provides the ground for regarding them as collectively forming an entity in its own right. The association relations which serve this purpose can be many and varied, e.g., a choir is a collection of people who come together on one or more occasions for the purpose of practising and performing certain types of musical work; a flock of birds is a collection of birds which act together in a coordinated way (e.g., feeding together, flying together as a group, roosting communally); a stamp collection is a collection of stamps assembled together in an organised way by a person; and so on. But it must not be too tight: although it may involve physical proximity or even physical contact, this does not normally extend to actual attachment, for then what is produced is not a collection but a compound object, of which the constituent components are individuals. In general, spatial proximity is not enough; the behaviour of the individuals is frequently important, so the collection is defined not just in terms of how it is demarcated from its environment but also in relation to the processes in which it participates [12]. A preliminary informal discussion of many of the factors that need to be considered in formulating a theory of collective association relations is given in [1]. The full formal characterisation of such a theory is work for the future.

References

1. Wood, Z.M., Galton, A.P.: A taxonomy of collective phenomena. Applied Ontology **4**(3-4) (2009) 267–292
2. Fine, K.: Compounds and aggregates. Noûs **28**(2) (1994) 137–158
3. Doepke, F.: Spatially coinciding objects. Ratio **24** (1982) 45–60
4. Baker, L.R.: The Metaphysics of Everyday Life. Cambridge University Press, Cambridge, UK (2007)
5. Thomson, J.J.: The statue and the clay. Noûs **32**(2) (1998) 149–173
6. Grenon, P.: BFO in a nutshell: A bi-categorial axiomatization of BFO and comparison with DOLCE. Technical Report 06/2003, IFOMIS, Universität Leipzig (December 2003)
7. Heller, M.: The Ontology of Physical Objects: Four-dimensional Hunks of Matter. Cambridge University Press, Cambridge (1990)
8. Sider, T.: Four-dimensionalism: An ontology of persistence and time. Oxford University Press, Oxford, UK (2001)
9. Gallois, A.: Occasions of Identity. Clarendon Press, Oxford (1998)
10. Winston, M.E., Chaffin, R., Herrman, D.: A taxonomy of part-whole relations. Cognitive Science **11** (1987) 417–444
11. Simons, P.: Parts: a Study in Ontology. Clarendon Press, Oxford, UK (1987)
12. Galton, A.P., Mizoguchi, R.: The water falls but the waterfall does not fall: New perspectives on objects, processes and events. Applied Ontology **4**(2) (2009) 71–107