Prolegomena to an Ontology of Shape

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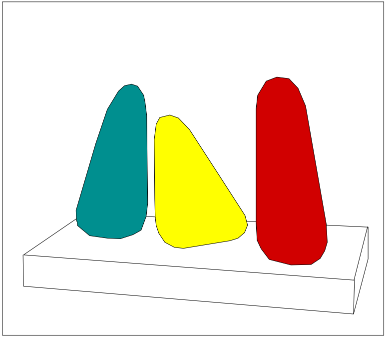
Physical Shape



Mathematical Shape

What things have shapes?

- Material objects, including
  - Chunks of matter
  - Organisms
  - Assemblies
- Non-material physical objects, including
  - Holes
  - Faces
  - Edges
  - Shadows
- Aggregates, collectives, etc.
- Abstract objects, such as
  - Geometrical figures

Talking About Shapes

- The shape of X
- X has such-and-such a shape
- X and Y have the same shape
- X is shaped like a Y
- X is Y-shaped

Talking About Shapes

- The shape of X
- X has such-and-such a shape
- X and Y have the same shape
- X is shaped like a Y
- X is Y-shaped
- The shape of X at time t
- X has such-and-such a shape at time t
- X and Y have the same shape at time t
- X changes shape between times t_1 and t_2

Shape as Property

circular
triangular
spherical
cylindrical
rectangular
square
oblong
heart-shaped
pear-shaped

Shape as Thing

circle
triangle
sphere
cylinder
rectangle
square
oblong
heart-shape
pear-shape

Which is logically / ontologically prior?

- **Shape as property**
  Logical analysis uses *shape predicates* such as $\mathit{Square}(x)$, $\mathit{Circular}(y)$.
  For generalising over shapes we must quantify over properties (second-order logic).

- **Shape as thing**
  Logical analysis uses *shape terms* to *reify* shape properties. Objects are related to their shapes by means of a predicate *HasShape*, e.g., $\mathit{HasShape}(x, \mathit{square})$, $\mathit{HasShape}(x, \mathit{circle})$.

  Ontologically, shapes are *generically dependent* entities (cf., information).

x and y have the same shape at t

- Shape as property:

$$\forall \Phi(\mathit{ShapeProperty}(\Phi) \rightarrow (\Phi(x,t) \leftrightarrow \Phi(y,t)))$$

- Shape as thing:

$$\forall s(\mathit{HasShape}(x,s,t) \leftrightarrow \mathit{HasShape}(y,s,t))$$

x and y have the same shape at t

- Shape as property:

$$\forall\Phi(ShapeProperty(\Phi) \rightarrow (\Phi(x,t) \leftrightarrow \Phi(y,t)))$$

- Shape as thing:

$$\forall s(HasShape(x,s,t) \leftrightarrow HasShape(y,s,t))$$

x changed shape between t_1 and t_2

- Shape as property:

$$\exists\Phi_1\exists\Phi_2(ShapeProperty(\Phi_1) \wedge ShapeProperty(\Phi_2) \wedge \\ \Phi_1(x,t_1) \wedge \Phi_2(x,t_2) \wedge \neg\Phi_1(x,t_2) \wedge \neg\Phi_2(x,t_1))$$

- Shape as thing:

$$\exists s_1\exists s_2(HasShape(x,s_1,t_1) \wedge HasShape(x,s_2,t_2) \wedge \\ \neg HasShape(x,s_2,t_1) \wedge \neg HasShape(x,s_1,t_2))$$

The view from modern ontology: BFO and DOLCE

Shape is *specifically dependent* on its bearer. Different bearers cannot have the same shape, but their separate shapes may have the same *value*.

The shape of x is $shape(x)$, which obeys the rule

$$\forall x \forall y (shape(x) = shape(y) \rightarrow x = y).$$

The values assumed by shapes are *shape qualia*, which collectively constitute *shape space*.

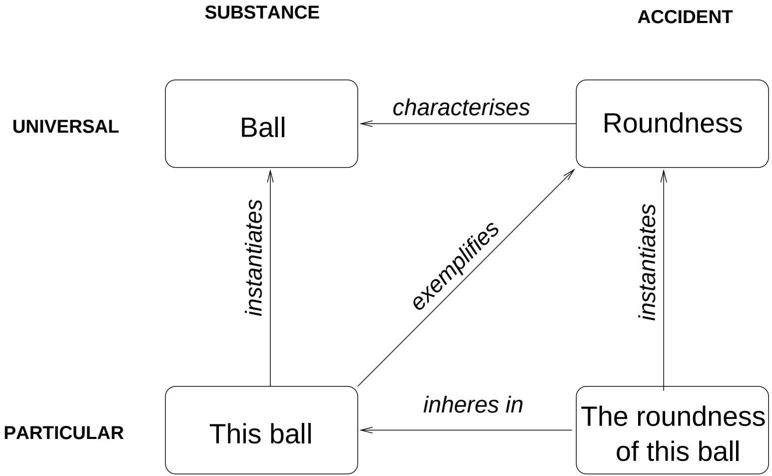
- **x and y have the same shape at t**

$$value(shape(x), t) = value(shape(y), t)$$

- **x changed shape between t_1 and t_2**

$$value(shape(x), t_1) \neq value(shape(x), t_2)$$

Aristotle's Four-Category Ontology (The Ontological Square)



The Primacy of "Same Shape" over "Shape"

Claim: The commonest (only?) way of describing the shape of something is by comparison with something else whose shape is assumed known:

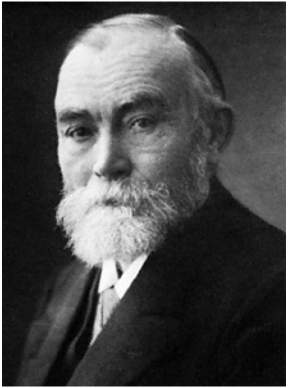
- "The table is square" — the table[-top] has the same (or sufficiently similar) shape as a certain geometrical construction.
- "The leaf is egg-shaped" — the leaf has the same (or sufficiently similar) shape as [the outline of] an egg.

Gottlob Frege (1848–1925)

Die Grundlagen der Arithmetik, 1884

(The Foundations of Arithmetic)

Frege drew attention to a group of concepts X for which the notion of an X is logically dependent on the notion of a relation "has the same X as" which can be defined without reference to X itself.



Examples: Number, Direction, Shape

Example 1: Number

Frege: *die Anzahl, welche dem Begriffe* F *zukommt* = *der Umfang des Begriffes "gleichzahlig dem Begriffe* F".

(the number of Fs = the extension of the concept "Has the same number as the Fs")

In terms of sets:

Set S has the same number as set S' if and only if there is a bijection between the elements of S and the elements of S'.

The number of elements in S	=	the set of all sets with the same number of elements as S

Example 2: Direction

"has the same direction as" $=$ "is parallel to"

the direction of line $L =$ the set of all lines parallel to L.

Example 3: Shape

"has the same shape as" $=$ "is geometrically similar to"

the shape of figure $F =$ the set of all figures similar to F

In general

Definitions like this work so long as:

- A domain of "objects" $\mathcal{Z}$ is established for the relation "has the same X as" to be defined on.
- Within the domain $\mathcal{Z}$, "has the same X as" can be defined as an equivalence relation.

Then we can say:

the X of $y \in \mathcal{Z}$ $=$ the set of all elements of $\mathcal{Z}$ that have the same X as y

"Same shape" for geometrical figures

- A geometrical figure is a set of points in $\mathbb{R}^n$.
- Write $\Delta(p, q)$ for the distance between points $p, q \in \mathbb{R}^n$.
- **Definition of geometric similarity between figures in Euclidean space:**

> *$X, Y \subseteq \mathbb{R}^n$ are geometrically similar if and only if there is a bijection $\phi : X \to Y$ such that, for some constant $\kappa \in \mathbb{R}^+$, the following relation holds:*
>
> $$\forall x, x' \in X : \Delta(\phi(x), \phi(x')) = \kappa \Delta(x, x').$$

- Thus defined, "geometrically similar" is an equivalence relation and therefore can be used as the definition of "has the same shape as".

Mathematical vs Physical Distance

- In $\mathbb{R}^n$, the notion of distance is unproblematic because numbers, i.e., elements of $\mathbb{R}$, are already built into the definition of the elements of the space.
- But physical space does not come already equipped with numbers.
- Assigment of numbers to physical space has to be accomplished by the physical act of *measurement*.
- But measurements always have finite precision.
- The definition of similarity has to be modified to take this into account.

Suppose

- we wish to measure distances between points within some object P of volume v.
- the smallest distance we can distinguish is h (our measurement process has "resolution h").

Then

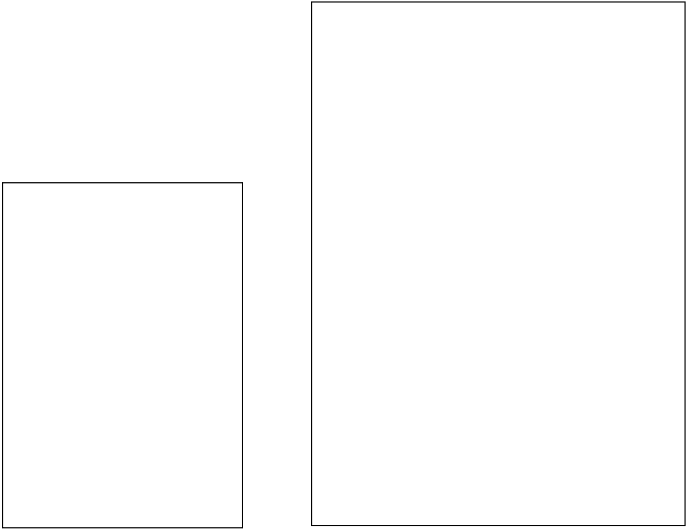
- Within the physical space occupied by P we can distinguish a set $S_h(P)$ containing some $n \approx v/h^3$ points.
- To each pair x, y of these points we can assign a distance $\Delta_h(x, y) = kh$ (where $k \in \mathbb{Z}$).

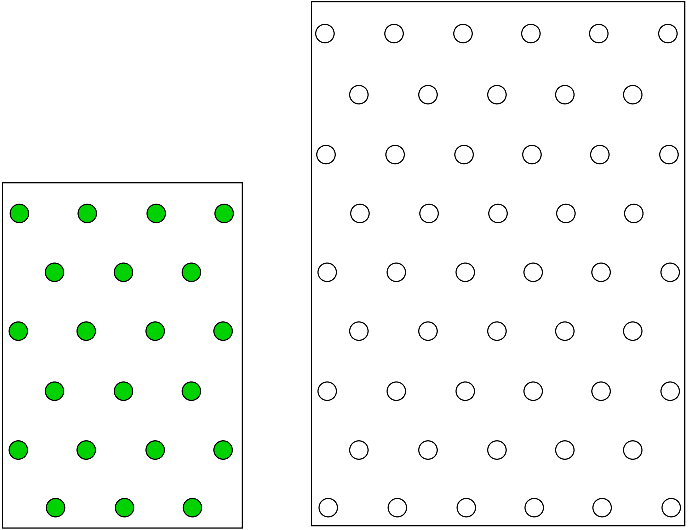
Given this, how do we compare distances within two different shapes in order to set up a similarity relation between them?

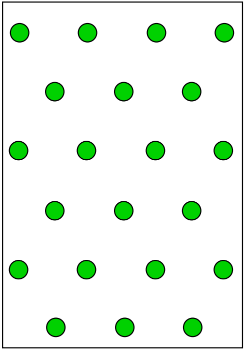
Definition of "same shape" for physical objects:

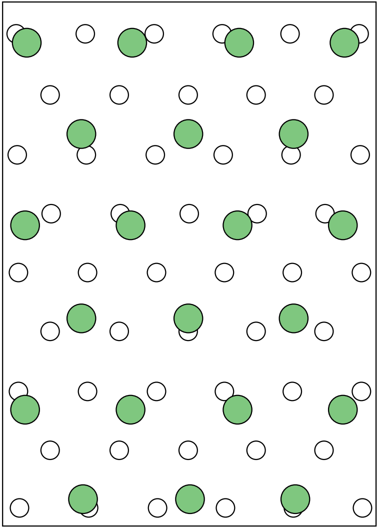
Physical objects P and Q (where Q is at least as big as P) have the same shape, at resolution h, if, for some constant $\kappa \geq 1$, the set $S_h(P)$ of points discernible in P at resolution h can be mapped into the set $S_h(Q)$ of such points of Q by means of an injective mapping ϕ, such that the following relations hold:

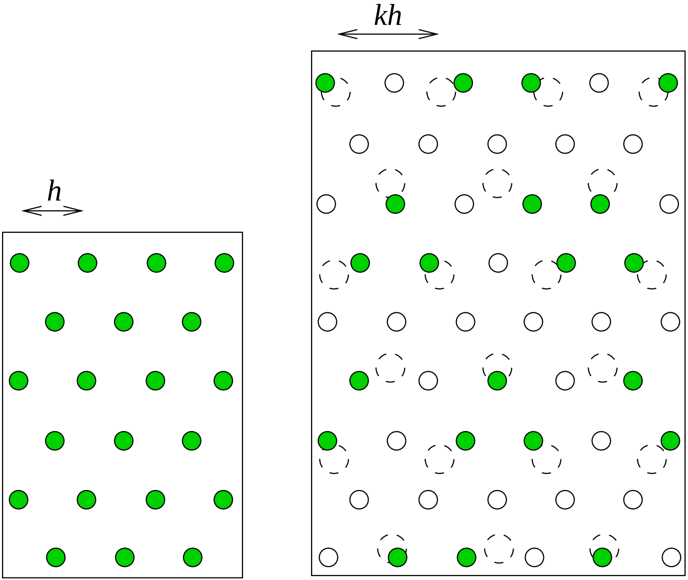
1. $\forall x, y \in S_h(P).\, |\Delta_h(\phi(x), \phi(y)) - \kappa\Delta_h(x, y)| \leq h$
2. $\forall x \in S_h(Q).\, \exists y \in S_h(P).\, \Delta_h(x, \phi(y)) \leq \kappa h$









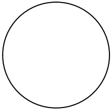

h
kh

Some observations

- Two objects may have the same shape at resolution h but different shapes at some resolution $h' < h$.
- Therefore, under the Fregean construction, the shape of an object would have to be a function of the resolution at which it is considered.
- But in fact the Fregean construction cannot be accomplished in this case, since "having the same shape at resolution h" is not an equivalence relation.[1]
- Therefore the notion of "exact shape" cannot be applied to physical objects

[1] It is a relation of indiscernibility, not of identity.

Comparing physical and geometrical shapes



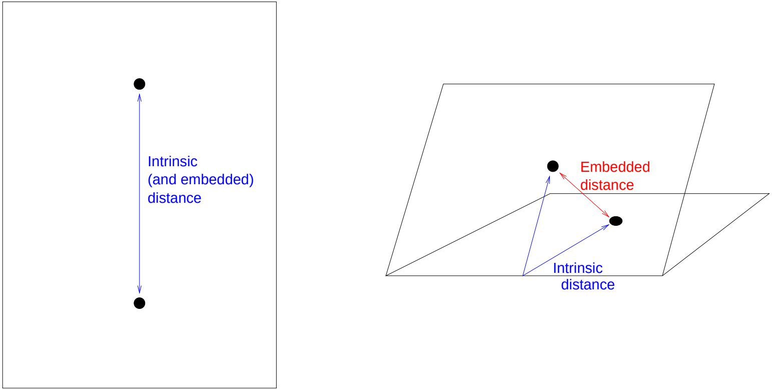
Lake Manicouagan is approximately circular: at some resolution, it has the same shape as a perfect geometrical circle. Neither of our "same shape" definitions can handle this. We need another one!

Definition of a physical object's having the "same shape" as a geometrical object

At resolution h, a physical object P has the same shape as a geometrical object Q if there is an injective mapping ϕ from the set of points $S_h(P)$ discernible in P at resolution h into the set of points in Q such that, for some constant $\kappa > 0$:

1. $\forall x, y \in S_h(P).\, \Delta(\phi(x), \phi(y)) = \kappa\Delta_h(x, y)$
2. $\forall x \in Q.\, \exists y \in S_h(P).\, \Delta(x, \phi(y)) \leq \kappa h.$

Instrinsic *vs* Embedded Distance



Intrinsic *vs* Embedded Distance

Given a geometrical object P embedded in a space S, the *P-intrinsic distance* between two points x, y in P is

$$\Delta_P(x,y) \quad = \quad \text{the length of the shortest path between } x \text{ and } y \text{ which lies wholly within } P$$

For Physical objects, as before, we modify this to take resolution into account, writing $\Delta_{P,h}(x,y)$ for the P-intrinsic distance between x and y at resolution h.

Intrinsic distance is contrasted with the *S-embedded distance* $\Delta(x,y)$ (or $\Delta_h(x,y)$) we used earlier.

Definition of "same intrinsic shape" for physical objects

Physical objects P and Q (where Q is at least as big as P) have the same intrinsic shape, at resolution h, if, for some constant $\kappa \geq 1$, the set $S_h(P)$ of points discernible in P at resolution h can be mapped into the set $S_h(Q)$ of such points of Q by means of an injection ϕ, such that the following relations hold:

1. $\forall x, y \in S_h(P). \, |\Delta_{Q,h}(\phi(x), \phi(y)) - \kappa \Delta_{P,h}(x, y)| \leq h$
2. $\forall x \in S_h(Q). \, \exists y \in S_h(P). \, \Delta_{Q,h}(x, \phi(y)) \leq \kappa h$

Scope of “Instrinsic Shape”

For what class of objects is there a significant contrast between intrinsic and embedded shape?

Examples

- Sheets of paper
- Strands of wool
- Human bodies

Non-examples

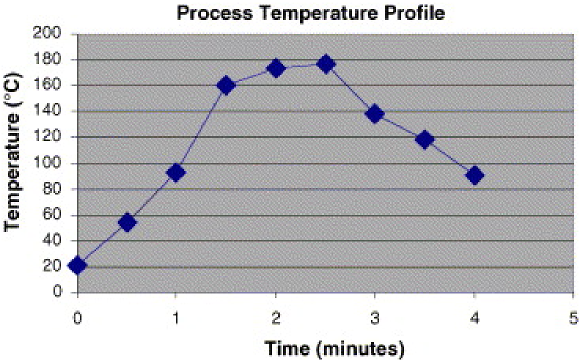
- Rigid objects
- Arbitrarily deformable objects (e.g., lumps of clay)

The positive examples are objects which have a “canonical” interrelationship of their parts which is preserved across the typical spatial transformations that the object undergoes.

Wanted: A more exact characterisation of the classes of objects for which the distinction between embedded and intrinsic shape applies.

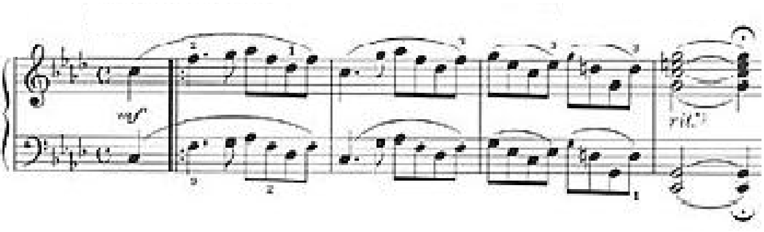
Extension: The "shape" of a process

Metaphorical "distance" leads to metaphorical "shape", e.g., the "shape" of a process (using distance in time, quality spaces)



(R. B. Prime, C. Michalski and C. M. Neag, 'Kinetic analysis of a fast reacting thermoset system', *Thermochimica Acta* **429** (2005) 213–217)

The Shape of a Musical Phrase



Johannes Brahms, Piano Quintet in F minor, Op.34

Conclusions

- The ontological status of shape is problematic because of its dependent character: shapes do not exist "in their own right", but only as qualities of objects.
- For geometrical figures, "same shape" is defined as geometrical similarity, providing a criterion of identity for geometrical shapes.
- For physical objects, we can only define "same shape at resolution h", which is not an equivalence relation and so does not supply a robust criterion of identity for shape.
- "Same shape" relations are based on a notion of "distance": either in the embedding space, or within the object itself, leading to the notion of intrinsic shape.
- Metaphorical "distance" leads to metaphorical "shapes", e.g., temporal process profiles, the shape of a musical phrase.