Multidimensional Mereotopology

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Abstract

To support commonsense reasoning about space, we require a qualitative calculus of spatial entities and their relations. One requirement for such a calculus, which has not so far been satisfactorily addressed in the mereotopological literature, is that it should be able to handle regions of different dimensions. Regions of the same dimension should admit Boolean sum and product operations, but regions of different dimensions should not. In this paper we propose a topological model for regions of different dimensions, based on the idea that a region of positive codimension is a regular closed subset of the boundary of a region of the next higher dimension. To satisfy the requirements of the commonsense theory, it is required that regions of the same dimension in the model can be summed, and we show that this is always the case. We conclude with a discussion of the possible applicability of the technical results to commonsense spatial reasoning.

Keywords

Qualitative spatial reasoning, mereotopology, dimension, boundary

Introduction

It is generally agreed that commonsense reasoning about space should be supported by a *qualitative* calculus of spatial entities and their relations. There is also a broad consensus that the most basic qualitative attributes and relations of spatial entities are mereotopological, i.e., concerned with the relations of *parthood* and *contact* and other relations and attributes that may be derived from these, such as overlap, external connection, and the distinction between tangential and non-tangential parts. But mereotopology alone is not sufficient to handle the full range of qualitative concepts important for commonsense spatial reasoning. One such concept which I shall not discuss here is *convexity*; another, which forms the central topic of this paper, is *dimension*.

Qualitative spatial reasoning must engage with the concept of dimension if it is to do justice to our commonsense apprehension of space. Many everyday spatial concepts carry information about dimension: some examples are *line*, *area*, *volume*, *edge*, *corner*, and *surface*. Lower-dimensional entities may arise as idealisations under coarse granularity of what are really higher-dimensional, for example the conceptualisation of roads and rivers as line objects in GIS. But sometimes we seem to need the notion of a strictly one- or two-dimensional entity inhabiting three-dimensional space, for example the *portal* of Hayes' Ontology of Liquids (Hayes 1985), which is defined as 'a piece of surface which links two pieces of space and through which objects and material can pass'.

For the mereological component of spatial reasoning it is important to have operations by which 'new' spatial entities can be derived from old, notably Boolean-like operations of sum, product, and complement. In set theory these are modelled by union, intersection and complementation, which form a true Boolean algebra, but in mereology it is often felt that there should be no 'null' entity corresponding to the empty set, leading to calculi that are analogous to, but in many ways more complicated than, true Boolean algebras. An example of mereological sum applied to lower-dimensional entities would be the bringing together of a collection of linear river stretches to form a branching river-system.

If we are to handle entities of different dimensions within a unified theoretical framework then we need to determine how entities of different dimensions are related. We may broadly distinguish between *bottom-up* and *top-down* approaches, according as higher-dimensional entities are derived from lower, or vice versa. The standard mathematical point-set approach illustrates the bottom-up method: zero-dimensional points are taken as primitive, and lines, surfaces and solids are constructed as sets of points. If arbitrary point-sets are allowed to count as spatial entities, then we end up with an ontology far too rich for the purposes of commonsense reasoning: arbitrary point-sets can exhibit all manner of pathological behaviours, including extreme disconnection, fractal-type convolutions, and bizarre 'mixed dimension' entities. Thus in the bottom-up approach, the process of construction must be constrained in some way, for example by allowing only simplicial complexes.

The top-down procedure is complementary to the bottom-up: starting with solids—three-dimensional chunks—as primitive entities, we define surfaces, lines and points as sets of solids. (Roughly, a lower-dimensional entity is defined as the set of all solids which we want to regard as 'containing' that entity.) This approach was explored by de Laguna (1922) and Tarski (1956), who were motivated by the thought that the spatial entities which are in some sense the most 'real' are precisely the solid, three-dimensional objects in the world around us, and the three-dimensional regions that they do, or can, occupy. Lower-dimensional entities are conceived of as in some sense dependent on these, and the top-down approach affirms this dependence by actually deriving them from solids—notwithstanding the rather counter-intuitive flavour of the resulting characterisations.

In this paper I review a number of mereotopological schemes from the literature, focussing on whether and how they handle regions of different dimensions. I then propose a mathematical model within which we can define spatial regions of different dimensions in a way which does justice to the essential insight that lower-dimensional regions arise as parts of the boundaries of higher-dimensional regions. An important concept which will facilitate much of the discussion is that of *codimension*: by this is meant the number of dimensions by which a region falls short of the dimensionality of the space in which it is considered to be embedded. Thus a one-dimensional object embedded in three-dimensional space has a codimension of 2. By 'lower-dimensional' regions is meant regions of positive codimension.

Current approaches

Regional Connection Calculus

One of the best-known approaches to the logical codification of commonsense mereotopological theory is the Regional Connection Calculus (RCC) of (Randell, Cui, & Cohn 1992). In the basic RCC-8 formulation, there is a single non-logical primitive, the binary relation C, interpreted as 'connection' or 'contact'. Additional relations are defined in terms of C, notably the following:

Part
$\mathsf{P}(x,y) =_{def} \forall z(\mathsf{C}(z,x) \rightarrow \mathsf{C}(z,y))$
Proper part
$\mathsf{PP}(x,y) =_{def} \mathsf{P}(x,y) \wedge \neg\mathsf{P}(y,x)$
Overlap
$\mathsf{O}(x,y) =_{def} \exists z(\mathsf{P}(z,x) \wedge \mathsf{P}(z,y))$
External connection
$\mathsf{EC}(x,y) =_{def} \mathsf{C}(x,y) \wedge \neg\mathsf{O}(x,y)$
Tangential proper part
$\mathsf{TPP}(x,y) =_{def} \mathsf{PP}(x,y) \wedge \exists z(\mathsf{EC}(z,x) \wedge \mathsf{EC}(z,y))$
Non-tangential proper part
$\mathsf{NTPP}(x,y) =_{def} \mathsf{PP}(x,y) \wedge \neg\mathsf{TPP}(x,y)$

A key axiom of RCC is $\forall x \exists y \mathsf{NTPP}(y,x)$, which says that every region has a non-tangential proper part. The motivation for this axiom is to ensure that space is not discrete, but it also succeeds in ruling out regions of positive codimension.

To see why, consider in 2D space a curve segment L, with a proper part P which does not extend to either of the extremities of L (Figure 1). For P to be NTPP to L, there should be no region simultaneously EC to both L and P. But area A in the diagram is exactly such a region, which means that P is TPP to L. The same reasoning would apply to *any* proper part of L, from which we conclude that L has no non-tangential proper parts, thereby contradicting the axiom.

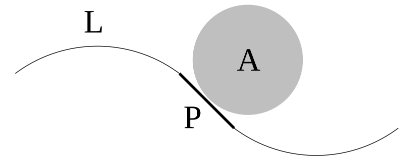

Figure 1: Proper parthood for regions of positive codimension

To avoid this conclusion, we need to interpret RCC so that A is not, after all, EC to P. Regions are EC so long as they are connected but do not overlap, so we need A and P to either overlap or be disconnected. In a point-set topological interpretation in which regions are connected if their closures are non-disjoint, A is certainly connected to P. If A is open, it does not overlap P, whereas if it is closed it has a point in common with P—but this is only overlap if a point counts as a region, in which case the notion of external connection disappears entirely, so *all* proper parts are non-tangential, contrary to the spirit of RCC. Thus while it is possible in principle to interpret RCC in such a way that regions of positive codimension can be accommodated, to do so would deprive RCC of some of its expressive power, since several of the defined predicates become null. Thus RCC is fundamentally antagonistic to regions of positive codimension.

Intersection matrices

Independently of RCC, Egenhofer introduced a method of capturing certain mereotopological relations between regions by means of matrices which record the nature of the intersections between salient parts of the regions (Egenhofer 1989; 1991). An example is the *9-intersection matrix*, defined for the regions X and Y as

$$\begin{pmatrix} \mathsf{b}(X)\cap\mathsf{b}(Y) & \mathsf{b}(X)\cap\mathsf{i}(Y) & \mathsf{b}(X)\cap\mathsf{c}(Y) \\ \mathsf{i}(X)\cap\mathsf{b}(Y) & \mathsf{i}(X)\cap\mathsf{i}(Y) & \mathsf{i}(X)\cap\mathsf{c}(Y) \\ \mathsf{c}(X)\cap\mathsf{b}(Y) & \mathsf{c}(X)\cap\mathsf{i}(Y) & \mathsf{c}(X)\cap\mathsf{c}(Y) \end{pmatrix}$$

where $\mathsf{b}(X)$, $\mathsf{i}(X)$, and $\mathsf{c}(X)$ are respectively the boundary, interior, and complement of X. In this context, these notions must be interpreted as follows. The complement is always understood with respect to the embedding space, but the interpretation of the other terms depends on the dimension of X. For example, if a line segment S in three-dimensional space is modelled as a subset of $\mathbb{R}^3$ with the usual topology, then the boundary of S is $\partial(S) = S$ and the interior of S is $int(S) = \emptyset$. But to use the 9-intersection matrix, we require $\mathsf{b}(S)$ to consist of just the two end-points of S, and $\mathsf{i}(S)$ to be $S \setminus \partial(S)$. Thus we must think of S as lying in some one-dimensional subset L of $\mathbb{R}^3$, and use the boundary and interior operations defined in the subspace topology induced on L from the topology on $\mathbb{R}^3$.[1] We will make similar use of subspace topologies in the model to be presented below.

With this understanding of boundary, interior and complement, the 9-intersection matrix for the regions A and P in Figure 1 is

$$\begin{pmatrix} \emptyset & \neg\emptyset & \neg\emptyset \\ \emptyset & \emptyset & \neg\emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Each entry indicates whether the corresponding intersection is empty or not. For example, the $\neg\emptyset$ appearing as the second item in the top row indicates that the boundary of A has non-empty intersection with the interior of P—the intersection consisting, in this instance, of the single point at which P is tangent to A. Since the topological relationship signified by this matrix cannot hold between regions of co-dimension zero,[2] this matrix does not correspond to any of the RCC-definable relations, all of which *can* be instantiated with regions of co-dimension zero. The full set of allowable 9-intersection matrices covers all the possible relations amongst regions, including regions of positive codimension, but is not able to discriminate all such relations uniquely. In particular, the 9-intersection matrix alone is insufficient to determine the dimensionality of the regions—in our example, the area A could be replaced by a line meeting P at the point of tangency with A, and this relationship between two line segments has the same 9-intersection as the relation portrayed between an area and a line.

What is a region?

As remarked above, if we model regions and other spatial entities by means of sets of points, then we should not admit arbitrary sets of points. Many different ways of restricting the class of point-sets that are to count as spatial entities have been proposed. Here we briefly review two of them.

Regular sets

A number of authors such as (Asher & Vieu 1995) have advocated *regular sets* as suitable models for spatial regions. In topology, an open set is regular if it is the interior of its closure; a closed set is regular if it is the closure of its interior. There seem to be sound commonsense reasons for excluding non-regular sets. A non-regular open set, for example, can have a line-like 'crack' running through its interior, consisting of boundary points which, instead of separating the interior from the exterior in accordance with the commonsense notion of boundary, only separate one portion of the interior from another. The points along such a crack will be included in the interior of the closure of the set, which is therefore not equal to the set itself, making the latter non-regular. Similarly, a non-regular closed set can have linear 'spikes' consisting of boundary points which only separate parts of the exterior; these do not occur in the closure of the interior of the set, which is therefore a proper subset of the set itself, again making the latter non-regular.

A relevant technical consideration is that the regular sets of a topology form a Boolean algebra under suitably-defined operations of sum $(X+Y)$, product $(X\cdot Y)$ and complement $(-X)$. The definition of these operations will differ according as we are dealing with regular open or regular closed sets, as follows:

- For regular open sets,
  - $X + Y = int(cl(X \cup Y))$ (this is the *regular union*),
  - $X \cdot Y = X \cap Y$ (since the intersection of regular open sets is always regular open),
  - $-X = int(X^c)$ (where X^c is the set-theoretic complement).
- For regular closed sets,
  - $X + Y = X \cup Y$ (since the union of two regular closed sets is always regular),
  - $X \cdot Y = cl(int(X \cap Y))$ (this is the *regular intersection*),
  - $-X = cl(X^c)$.

One consequence of using regular sets is that it seems to limit one to regions of co-dimension zero, since no set of positive codimension can be regular, having empty interior. In discussing the Egenhofer system, we noted that the notions of boundary and interior for regions of positive co-dimension have to be understood relative to a subspace topology, and later we shall use this idea to allow us to regard even sets of positive codimension as being in some sense regular.

Polygons and polyhedra

In a number of publications, Pratt-Hartmann and colleagues have pointed out that, attractive though regular sets might seem as a technical counterpart of commonsense spatial regions, they include some rather badly-behaved examples: e.g., in $\mathbb{R}^2$, regions whose boundary includes a portion of the well-known 'pathological' curve $y = \sin(x^{-1})$ in the neighbourhood of $x = 0$. To remedy this, they proposed the restriction, in two dimensions, to *polygonal* (Pratt & Lemon 1997; Pratt & Schoop 1997), or, in three dimensions, to *polyhedral* (Pratt-Hartmann & Schoop 2002) regions. In $\mathbb{R}^2$, a *half-plane* is that portion of space lying to one side of some (infinite) straight line; a *basic polygon* is the intersection of finitely many half-planes; and a *polygon* is the regular union of any finite set of basic polygons. A polyhedron in $\mathbb{R}^3$ is defined analogously.

From a commonsense point of view, the restriction to polygons or polyhedra is attractive because (a) it avoids the pathologies associated with an unrestricted diet of regular sets while retaining their Boolean algebra structure, and (b) arbitrary regular regions can be approximated as closely as desired by polygons or polyhedra—indeed, just such approximation is standardly used in GIS, where areas are represented as polygons and volumes as polyhedra. However, none of Pratt-Hartmann's mereotopologies allows one to handle regions of different dimensions within one and the same system.

[1] In the subspace topology on L, the open sets are those sets of the form $L \cap O$, where O is open in $\mathbb{R}^3$, and similarly for closed sets.

[2] For such regions, if ∂A intersects $int(P)$, then $int(A)$ must also intersect $int(P)$.

Axiomatic approaches

Desiderata for regions of different dimensionalities

What should we be able to do with regions of different dimension? One reasonable requirement is that while we might allow arbitrary Boolean combinations of regions of the *same* dimension, combining regions of different dimension should not be admitted. The intuition here is that length, area, and volume are in some sense *sui generis*: it does not make sense to combine an element which has area with an element which has only length and to call the resulting element a region. If this is accepted, then we clearly cannot understand regions as sets of points with unrestricted possibilities of combination.

On the other hand, we should not rule out the possibility of describing *relations* between regions of different dimension, exactly as is allowed in the Egenhofer's system and related systems such as the Calculus-Based Method of (Clementini, di Felice, & van Oosterom 1993). We need to be able to say that a path crosses an area, for example, or follows its boundary.

One way to approach these requirements would be to try to express them in some appropriately tailored logical language. In this section we examine some existing proposals for such a language. An alternative approach, which we follow later, is to try to construct an explicit mathematical model for regions of different dimensions which will exhibit the properties that we desire. Ultimately one would want to combine both approaches by ensuring that the mathematical model satisfies the logically-expressed requirements.

Gotts's INCH Calculus

Gotts (1996) proposed a logical language with a single primitive binary relation $\mathsf{INCH}(x, y)$, to be read 'x includes a chunk of y'. The variables of the language range over 'extents', which Gotts defines as 'closed sets of points of uniform dimensionality, with a locally finite triangulation, within a locally Euclidean space'. He allows different extents to have different dimensions, but he does not allow extents of mixed dimension. A part of an extent x having the same dimension as x is called a 'chunk' of x, and the intended meaning of $\mathsf{INCH}(x, y)$ is that some chunk of y lies wholly within x.

Gotts sets out a provisional list of ten axioms for INCH which he claims 'express a significant portion of our knowledge of commonsense topology'. The axioms make use of a number of additional predicates all defined in terms of INCH. For example, x has dimensionality at least that of y so long as it INCHes something which INCHes y; equidimensional regions are then those each of which has dimensionality at least that of the other.

Gotts notes that we cannot apply Boolean operations to arbitrary pairs of extents, since this may result in extents of mixed dimensionality, which are not countenanced by the INCH-calculus. Instead he restricts Boolean operations to pairs of equidimensional extents, which are controlled by two axioms guaranteeing the existence of Boolean sums and differences in these cases. The sum of x and y can be formed so long as x and y are equidimensional, and is defined to be the unique extent which INCHes all and only those extents INCHed by at least one of x and y. Likewise the difference between x and y is the extent which INCHes precisely those extents which are INCHed by some chunk of x which does not overlap y. The product can be defined in the usual way in terms of sum and difference. Gotts concludes that a set of equidimensional extents forms a distributive lattice under the sum and product operations.

Galton's 'Taking dimension seriously'

Independently of Gotts's work, I proposed in (Galton 1996) an axiomatic system for multidimensional mereotopology using primitives for 'part' (P) and 'boundary' (B). The mereological component differed from classical extensional mereology (Simons 1987) in not allowing sums of arbitrary pairs of regions: instead, we may form the sum of a set of regions only if there is some encompassing region of which all the regions in the set are parts. Regions which form part of some larger region (and thus are summable) are said to be equidimensional. In effect, this produces a *layered* mereotopology in the sense of (Donnelly & Smith 2003), with a layer for each dimensionality (a layer of zero-dimensional entities, a layer of one-dimensional entities, a layer of two-dimensional entities, and so on). Within each layer, Boolean sum, product and difference operations can be performed, but no such operations are possible between layers.[3]

The topological component of the mereotopology handles relations between layers, expressed in terms of the primitive B, where $B(x, y)$ means that x bounds y. To say that x is of lower dimension than y is to say that something equidimensional to x bounds something equidimensional to y. This in turn allows us to say that x is of the *next lower* dimension than y, viz., x is of lower dimension than y, but is not of lower dimension than anything which is itself of lower dimension than y. It follows from the axioms and definitions that x is of the next lower dimension than y if and only if it is equidimensional to the boundary of something equidimensional to y.

Smith and Varzi (1997) made similar use of a bounding predicate B, but they defined the boundary of x to be the sum of everything which bounds x: $b(x) = \sigma z(B(z, x))$. This would not be allowed in Gotts's or Galton's systems, since these outlaw summation of regions of different dimensionality. Instead, the boundary of x must be defined as the sum of those regions of the next dimension lower than x, which bound x.

Mathematical modelling

In (Galton 2000) I suggested a way of defining regular open regions of different dimensions in $\mathbb{R}^3$. This can be easily generalised to an embedding space of any dimension. The idea is to take regular open sets in $\mathbb{R}^n$ (the embedding space) as the regions of dimension n (forming the collection $\mathcal{R}^n_n$), and then to take regular open subsets of their boundaries as regions of dimension $n-1$ ($\mathcal{R}^n_{n-1}$), and extend this process

[3] The term 'layers' was not explicitly used in (Galton 1996).

recursively until we reach regions of dimension 0 (which are just finite point-sets).

- The collection $\mathcal{R}^n_n$ consists of *regular open sets* in $\mathbb{R}^n$.
- Given the collection $\mathcal{R}^n_r$ ($1 \leq r \leq n$), the set $\mathcal{R}^n_{r-1}$ consists of those sets which are regular open in the subspace topology defined on the topological boundary of some member of $\mathcal{R}^n_r$—in other words, an element of $\mathcal{R}^n_{r-1}$ is a set of the form $C \cap \partial B$, where $C \in \mathcal{R}^n_n$, $B \in \mathcal{R}^n_r$.[4]

Here $\mathcal{R}^n_r$ consists of regions of dimension r embedded in a space of dimension n. As already pointed out in (Galton 2000) there is a serious problem with this strategy, which is that it does not allow us to form even some quite simple sums of regions of positive codimension. At first, one might imagine that this arises from our liberally allowing arbitrary regular open sets as regions, but the same problems arise if, taking our cue from Ian Pratt-Hartmann's work on polygonal and polyhedral mereotopologies, we apply the same idea and, by analogy with the $\mathcal{R}^n_r$ series, introduce the $\mathcal{P}^n_r$ series as follows:

- The set $\mathcal{P}^n_n$ consists of *regular open polytopes* in $\mathbb{R}^n$.
- The set $\mathcal{P}^n_{r-1}$ ($1 \leq r \leq n$) consists of all sets of the form $C \cap \partial B$, where $C \in \mathcal{P}^n_n$, $B \in \mathcal{P}^n_r$.

It turns out that, even with this model, we cannot always define the sum of two elements of $\mathcal{P}^n_r$, where $r < n$, in a way that accords with our commonsense requirements. To illustrate in $\mathcal{P}^2_1$, it seems reasonable that we should form the sum of the two regions

$$\begin{aligned} R_1 &= \{(x,0) \mid -1 < x < 1\} \\ R_2 &= \{(0,y) \mid 0 < y < 1\} \end{aligned}$$

leading to a $\perp$-shaped region $R = R_1 \cup R_2$. Now R_1 and R_2 are certainly in $\mathcal{P}^2_1$; the problem is that R is not. In order for R to be in $\mathcal{P}^2_1$, it must be $C \cup \partial B$, where C and B are in $\mathcal{P}^2_2$ (i.e., regular open polygons). Now consider the point $(0,0)$, which is in R, and therefore in both C and ∂B. Since C is open, $(0,0)$ is an interior point of C. Consider a small circle of radius ϵ inscribed about $(0,0)$. As you go round the circle you cross ∂B some number of times: certainly at $(0,\epsilon)$, $(\epsilon,0)$, and $(-\epsilon,0)$. Each time you cross it, you move into or out of B. It follows that in a complete circuit you must cross ∂B an even number of times. Hence there is at least one more crossing in addition to the three already listed. If ϵ is made small enough, the circle falls entirely inside C, and hence the extra border crossing is in $C \cap \partial B$. We have proved that *any set of the form $C \cap \partial B$ (where C and B are regular open polygons) which contains R must also contain points not in R.* It follows that R is not of the form $C \cap \partial B$. See the left hand illustration in Figure 2, in which B is shown shaded, C is indicated by the solid outline, and $C \cap \partial B$ is indicated by the bold lines.[5]

[4] I use the following mnemonic: C is a region which *contains* the region we are defining, and B is a region which is *bounded* by it.

[5] I maintain these diagrammatic conventions in all subsequent figures.

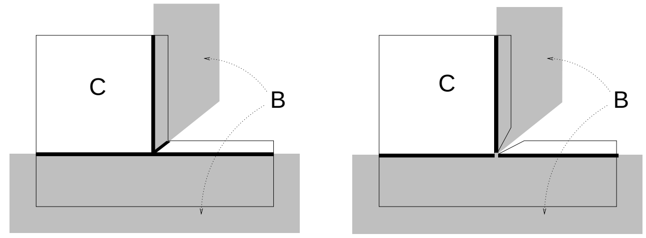

Figure 2: The '$\perp$' shape is not in $\mathcal{P}^2_1$

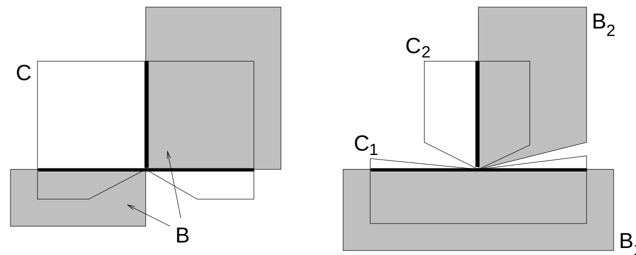

Figure 3: The '$\perp$' shape is in $\overline{\mathcal{P}}^2_1$

The right-hand illustration shows that if we draw the boundary of C in so that it does not include any of the fourth branch of ∂B, then $C \cap \partial B$ does not contain $(0,0)$ (since this is now on ∂C, and therefore not in C). More generally, it is clear that any set of the form $C \cap \partial B$ contained in R must be a *proper* subset of R. Thus once again we do not obtain R as a member of $\mathcal{P}^2_1$.

The root cause of the problem is that regions are modelled as regular *open* sets. A central claim of the present paper is that if instead we use regular closed sets then the problem no longer arises. In this case, however, we must take account of the fact that for closed sets regular intersection is not the same as set-theoretical intersection. Moreover, since it is defined with respect to a topology, and we are working with a number of different topologies at once (i.e., not just the topology on $\mathbb{R}^n$ but also various subspace topologies induced by this), we need to specify the topology with respect to which any particular application of regular intersection is to be understood. To facilitate this, I shall introduce a special notation, as follows:

By $P \sqcap Q$ is meant $cl(int(P \cap Q))$, where the operations cl and int are performed with respect to the subspace topology on Q.

We now define the $\overline{\mathcal{P}}^n_i$ series as follows:[6]

- The set $\overline{\mathcal{P}}^n_n$ consists of *regular closed polytopes* in $\mathbb{R}^n$.
- The set $\overline{\mathcal{P}}^n_{i-1}$ ($1 \leq i \leq n$) consists of all sets of the form $C \sqcap \partial B$, where $C \in \overline{\mathcal{P}}^n_n$, $B \in \overline{\mathcal{P}}^n_i$.

Now consider Figure 3. In the left-hand figure, R is presented as $C \sqcap \partial B$, where C and B are both regular closed

[6] The use of $C \sqcap \partial B$ here rather than $C \cap \partial B$ is important in order to prevent $\overline{\mathcal{P}}^n_i$ containing regions of dimension less than i, which are closed but not regular in an i-dimensional topology.

sets in $\mathbb{R}^2$, and therefore in $\overline{\mathcal{P}}^2_2$. Hence R is in $\overline{\mathcal{P}}^2_1$. Its components R_1 and R_2 are also in $\overline{\mathcal{P}}^2_1$, and this is shown in the right-hand diagram. Here R_1 is presented as $C_1 \sqcap \partial B_1$, and R_2 as $C_2 \sqcap \partial B_2$, where B_1, B_2, C_1, and C_2 are all in $\overline{\mathcal{P}}^2_2$. Since these sets are closed, they all contain the point at the junction of the $\perp$-shape—it is in fact a boundary point of each set. This means that R can be expressed as $(C_1 \cup C_2) \sqcap \partial(B_1 \cup B_2)$, once again showing it to be in $\overline{\mathcal{P}}^2_1$.

The general result we wish to prove is

Theorem 1. For fixed $n > 0$, and for $r = 1, \ldots, n$, the set $\overline{\mathcal{P}}^n_r$ is closed under union, i.e., whenever $R_1, R_2 \in \overline{\mathcal{P}}^n_r$ we have $R_1 \cup R_2 \in \overline{\mathcal{P}}^n_r$.

The case $r = n$ is straightforward, since in this case we are dealing with $\overline{\mathcal{P}}^n_n$, the set of regular closed polytopes in $\mathbb{R}^n$, which is already known to be closed under union. The general theorem will be proved by induction on the codimension $n - r$; thus $r = n$ will be the base case, and for $r < n$ we need to derive the result for $\overline{\mathcal{P}}^n_r$ from that for $\overline{\mathcal{P}}^n_{r+1}$.

Specifically the minimum we require (corresponding to the left hand illustration in Figure 3) is

Lemma 1. For $r < n$, if $\overline{\mathcal{P}}^n_{r+1}$ is closed under union, then for any $R_1, R_2 \in \overline{\mathcal{P}}^n_r$ there are regions $B \in \overline{\mathcal{P}}^n_{r+1}, C \in \overline{\mathcal{P}}^n_n$ such that $R_1 \cup R_2 = C \sqcap \partial B$.

This is what we need to ensure that $\overline{\mathcal{P}}^n_i$ is closed under union and hence can be given the structure of a Boolean algebra.

The right-hand illustration in Figure 3 corresponds to a stronger result, exemplified in this case by the relationship

$$(C_1 \sqcap \partial B_1) \cup (C_2 \sqcap \partial B_2) = (C_1 \cup C_2) \sqcap \partial(B_1 \cup B_2).$$

This relationship does not hold in general: we had to choose B_1, B_2, C_1 and C_2 specially to make it hold in this case. If this situation is to be generally applicable, then we need the more stringent requirement represented by the following lemma.

Lemma 2. For $r < n$, if $\overline{\mathcal{P}}^n_{r+1}$ is closed under union, then for any $R_1, R_2 \in \overline{\mathcal{P}}^n_r$ there are regions $B_1, B_2 \in \overline{\mathcal{P}}^n_{r+1}$ and $C_1, C_2 \in \overline{\mathcal{P}}^n_n$ such that

$$\begin{aligned} R_1 &= C_1 \sqcap \partial B_1 \\ R_2 &= C_2 \sqcap \partial B_2 \\ R_1 \cup R_2 &= (C_1 \cup C_2) \sqcap \partial(B_1 \cup B_2) \end{aligned}$$

It is obvious that Lemma 2 implies Lemma 1, and that either of them can play the part of the induction step in proving Theorem 1.

In what follows, I shall outline the proof of Lemmas 1 and 2. The proof as presented is not fully rigorous, but I believe that it would be relatively straightforward (if tedious) to make it so. Although the proof is intended to apply in any number of dimensions, the illustrative example I shall use relates to the induction step from $\overline{\mathcal{P}}^2_2$ to $\overline{\mathcal{P}}^2_1$.

To appreciate the problem, consider Figure 4, in which the bold lines pick out a Y-shaped one-dimensional region $R \in \overline{\mathcal{P}}^2_1$ defined as

$$R = (C_1 \sqcap \partial B_1) \cup (C_2 \sqcap \partial B_2).$$

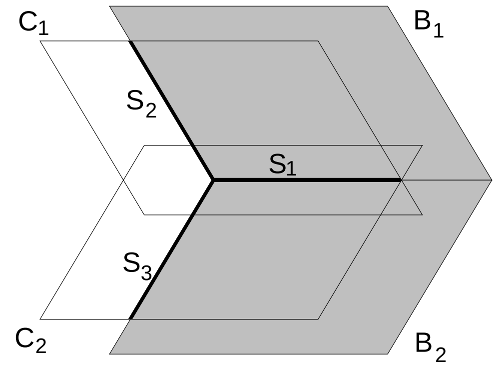

Figure 4: A problem case for the theorem?

We can put $R = R_1 \cup R_2$, where

$$\begin{aligned} R_1 &= S_1 \cup S_2 = C_1 \sqcap \partial B_1, \\ R_2 &= S_1 \cup S_3 = C_2 \sqcap \partial B_2. \end{aligned}$$

However,

$$(C_1 \cup C_2) \sqcap \partial(B_1 \cup B_2) = S_2 \cup S_3 \neq R,$$

since S_1 lies in the interior of $B_1 \cup B_2$. Therefore the regions B, C required by Lemma 1 cannot in this case be identified with $B_1 \cup B_2$ and $C_1 \cup C_2$.

The key to establishing the lemmas is provided by the decomposition of R into segments S_1, S_2, S_3. For $R_1, R_2 \in \overline{\mathcal{P}}^n_r$, we define a **segment** for $\{R_1, R_2\}$ to be a non-empty regular closed subset S of $R_1 \cup R_2$ such that

1. One of the following holds:
   - $int(S) \subseteq R_1 \setminus R_2$ (in which case S is called an R_1-segment);
   - $int(S) \subseteq R_2 \setminus R_1$ (an R_2-segment);
   - $int(S) \subseteq R_1 \cap R_2$ (a shared segment);
2. S is maximal with respect to condition 1, i.e., no S' such that $S \subset S' \subseteq R_1 \cup R_2$ satisfies condition 1.

It is clear that, given $\{R_1, R_2\}$, the union $R_1 \cup R_2$ can be uniquely decomposed into finitely many segments $S_1, S_2, \ldots, S_k$ such that

1. $R_1 \cup R_2 = S_1 \cup S_2 \cup \cdots \cup S_k$, and
2. For $i \neq j$, $S_i \cap S_j \subseteq \partial S_i \cap \partial S_j$.

That there are only finitely many segments is a consequence of the way R_1 and R_2 are ultimately derived from finitely many polytopes in $\mathbb{R}^n$. Condition 2 ensures that distinct segments can only overlap at their boundaries. (In Figure 4, S_2 is an R_1-segment, S_3 is an R_2 segment, and S_1 is a shared segment. The three segments meet at a single point—the junction of the 'Y'—on their shared boundary.)

The idea is to specify, for each segment, a containing region $C \in \overline{\mathcal{P}}^n_n$ and a bounded region $B \in \overline{\mathcal{P}}^n_{r+1}$ such that the segment itself is equal to $C \sqcap \partial B \in \overline{\mathcal{P}}^n_r$. Moreover, the bounded and containing regions for any given segment must be as disjoint as possible from those for other segments. This cannot be completely achieved, since some pairs of segments will meet at their boundaries; but we can at least ensure that the bounded and containing regions for one segment will only overlap those for a second segment at those shared boundary points where the two segments meet. I shall call this process *isolating* the segments. So long as the bounded and containing regions for the segments are kept apart in this way, we can then express the union of the segments (our target region) in terms of a bounded region which is the union of the bounded regions for the individual segments and a containing region which is the union of their containing regions.

Consider an R_1 segment S_i, where $R_1 = C_1 \sqcap \partial B_1$. For the containing region for S_i we will choose an element of $\overline{\mathcal{P}}^n_n$ which is contained in C_1 and contains S_i, and for the bounded region of S_i we will choose an element of $\overline{\mathcal{P}}^n_{r+1}$ which is contained in B_1 and whose boundary includes S_i. For an R_2 segment we do the same thing but using B_2 and C_2. For a shared segment, we could do either—let us simply adopt the convention of treating shared segments as if they were R_1 segments.

The exact requirements for isolating the segments are as follows:

1. For each segment S_i of $\{R_1, R_2\}$, we define a *containing region* $c(S_i) \in \overline{\mathcal{P}}^n_n$ and a *bounded region* $b(S_i) \in \overline{\mathcal{P}}^n_{r+1}$ such that $S_i = c(S_i) \sqcap \partial b(S_i)$.
2. If S_i is an R_1 or shared segment, then $c(S_i) \subseteq C_1$ and $b(S_i) \subseteq B_1$; but if it is an R_2 segment then $c(S_i) \subseteq C_2$ and $b(S_i) \subseteq B_2$.
3. For $i \neq j$, we have
   - $c(S_i) \cap c(S_j) \subseteq \partial S_i \cap \partial S_j$,
   - $b(S_i) \cap b(S_j) \subseteq \partial S_i \cap \partial S_j$,
   - $b(S_i) \cap c(S_j) \subseteq \partial S_i \cap \partial S_j$.

An essential part of proving Lemma 2 will be to establish that it is always possible to isolate the segments of $\{R_1, R_2\}$ in this way; I indicate at the end of this section how this is to be done. Meanwhile, assuming we can isolate the segments, let $R_1 = C_1 \sqcap \partial B_1$, $R_2 = C_2 \sqcap \partial B_2$, where $B_1, B_2 \in \overline{\mathcal{P}}^n_{r+1}$ and $C_1, C_2 \in \overline{\mathcal{P}}^n_n$. We need to find $C \in \overline{\mathcal{P}}^n_n, B \in \overline{\mathcal{P}}^n_{r+1}$ such that $R_1 \cup R_2 = C \sqcap \partial B$. Let

$$B = \bigcup_{i=1}^{k} b(S_i), \quad C = \bigcup_{i=1}^{k} c(S_i),$$

so

$$C \sqcap \partial B = \bigcup_{i=1}^{k} c(S_i) \sqcap \partial \bigcup_{i=1}^{k} b(S_i).$$

Note that here we are making use of the inductive hypothesis, that $\overline{\mathcal{P}}^n_{r+1}$—the set to which belong the regions $b(S_i)$ and $c(S_i)$—is already assumed to be closed under union.

Now, the regions $b(S_i)$ are of dimension $r+1$, and they only overlap with each other, if at all, inside regions of the form ∂S_i which are of dimension $r-1$. Hence their boundaries $\partial b(S_i)$ also only overlap in such regions. This being so, the boundary of their union is equal to the union of their boundaries, since the the union could only 'lose' a stretch of boundary if it were shared between two of the component regions making up the union (as S_1 is shared between B_1 and B_2 in Figure 4). Hence we can put

$$\begin{aligned} C \sqcap \partial B &= \bigcup_{i=1}^{k} c(S_i) \sqcap \bigcup_{i=1}^{k} \partial b(S_i) \\ &= \bigcup_{i=1}^{k} \bigcup_{j=1}^{k} c(S_i) \sqcap \partial b(S_j) \\ &= \bigcup_{i=1}^{k} c(S_i) \sqcap \partial b(S_i)) \\ &= \bigcup_{i=1}^{k} S_i = R \end{aligned}$$

as required for Lemma 1.[7]

Now let $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ be the sets of R_1-segments, R_2-segments, and shared segments respectively. If

$$I_1 = \{i \mid S_i \in \mathcal{S}_1 \cup \mathcal{S}_3\}, \quad I_2 = \{i \mid S_i \in \mathcal{S}_2 \cup \mathcal{S}_3\}$$

then if we put, for $i = 1, 2$,

$$C'_i = \bigcup_{j \in I_i} c(S_j), B'_i = \bigcup_{j \in I_i} b(S_j),$$

(note further use of the induction hypothesis) we have

$$\begin{aligned} R_1 &= C'_1 \sqcap \partial B'_1 \\ R_2 &= C'_2 \sqcap \partial B'_2 \\ R_1 \cup R_2 &= (C'_1 \cup C'_2) \sqcap \partial(B'_1 \cup B'_2) \end{aligned}$$

as required for Lemma 2.[8]

This process is illustrated in Figure 5. Here the outlines of regions B_1, B_2, C_1, C_2 from Figure 4 are indicated by dotted lines. The containing regions $c(S_i)$ for $i = 1, 2, 3$ are indicated by solid outlines, and the bounded regions $b(S_i)$ by shading. It can be seen that the segments S_i ($i = 1, 2, 3$) have been isolated in accordance with the requirements specified above, and that moreover

$$\begin{aligned} &R_1 \cup R_2 \\ &= S_1 \cup S_2 \cup S_3 \\ &= (c(S_1) \cup c(S_2) \cup c(S_3)) \sqcap \partial(b(S_1) \sqcap b(S_2) \sqcap b(S_3)). \end{aligned}$$

It remains to justify the assumption that it is always possible to isolate the segments of $\{R_1, R_2\}$ in the way required for the proof. Let S_i be an R_1-segment or a shared segment (what we do in these cases will apply, *mutatis mutandis*, to R_2-segments). Let $x \in int(S_i)$. We know that $x \notin S_j$ since S_j only meets S_i at its boundary. Since S_j is closed, this means that the distance from x to S_j is positive:

$$\min_{y \in S_j} d(x, y) > 0.$$

[7]The transition from the second to the third line is justified by the observation that when $i \neq j$, $c(S_i) \cap b(S_j) \subseteq \partial S_i \cap \partial S_j$, and hence $c(S_i) \sqcap \partial b(S_j) = \emptyset$.

[8]Note also that

$$R_1 \cdot R_2 = (C'_1 \cdot C'_2) \sqcap \partial(B'_1 \cdot B'_2),$$

giving the product of the regions as well.

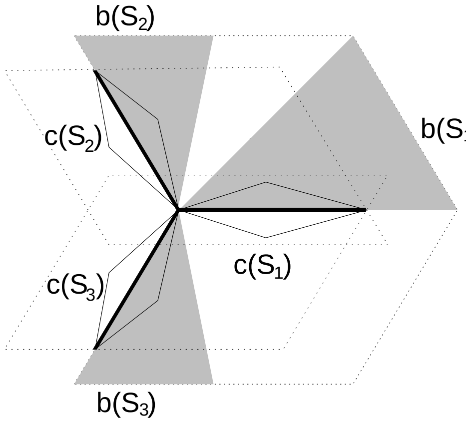

Figure 5: The problem solved!

Moreover, there are only finitely many segments S_j, which means that there is a minimum distance from x to any of the other segments, say

$$k(x) = \min_{j \neq i} \min_{y \in S_j} d(x, y).$$

Thus the closed n-sphere $\overline{B_n}(x, \frac{1}{4}k(x))$ of radius $\frac{1}{4}k(x)$ centred on x does not intersect any of the segments S_j where $j \neq i$.[9]

$$k(S_i) = cl\left(C_1 \cdot \bigcup_{x \in int(S_i)} \overline{B_n}(x, \frac{1}{4}k(x))\right).$$

This set contains S_i and is contained in C_1 and is of dimension n. Moreover, for $i \neq j$ we have $k(S_i) \cap k(S_j) \subseteq \partial S_i \cap \partial S_j$. Any element of $\overline{\mathcal{P}}^n_n$ such that $S_i \subset c(S_i) \subseteq k(S_i)$ will do for $c(S_i)$. That such an element can be found follows from the fact that any subset of $\mathbb{R}^n$ can be approximated arbitrarily closely by elements of $\overline{\mathcal{P}}^n_n$. This means that in fact there are infinitely many candidates satisfying the requirements for $c(S_i)$.

For $b(S_i)$ we employ a similar construction, suitably modified:

$$v(S_i) = cl\left(B_1 \cdot \bigcup_{x \in int(S_i)} \overline{B_{r+1}}(x, \frac{1}{3}k(x))\right).$$

We let $b(S_i)$ be an element of $\overline{\mathcal{P}}^n_{r+1}$ such that $S_1 \subseteq \partial b(S_i)$ and $b(S_i) \subseteq v(S_i)$. The reason we need a larger fraction ($\frac{1}{3}$) here than in the previous construction is as follows: we need to ensure that no part of the boundary of $b(S_i)$ falls within $c(S_i)$ except S_i itself, for otherwise $c(S_i) \sqcap \partial b(S_i)$ would include parts of $\partial b(S_i)$ in addition to S_i.

[9]The choice of $\frac{1}{4}$ here is arbitrary—any fraction less than $\frac{1}{2}$ would do. Similarly, the fraction $\frac{1}{3}$ in the construction for $b(S_i)$ given below could be replaced by any fraction between $\frac{1}{2}$ and the fraction chosen for the $c(S_i)$ construction.

Discussion

The main result of this paper was to establish that the $\overline{\mathcal{P}}^n_r$ series provides a suitable model for a multidimensional mereotopology in which regions of lower dimension are defined as regular closed subsets of the boundaries of regions of the next dimension up, and regions of the same dimension can be mereologically summed, but regions of different dimensions cannot, in accordance with the intuition that extension of each dimensionality is *sui generis*.

As a natural next step, we need to establish the relationship between the mathematical model and formal languages such as those proposed by Gotts (1996) and Galton (1996). We need a suitable language whose primitives can be interpreted in terms of the model, and to investigate the theory of the model as expressed in that language—in particular to look at possible axiomatisations and their metatheoretical properties.

Beyond this, it is important also to consider the applicability of the theory. As noted in the introduction, there is a difference between lower-dimensional entities which arise as representations under coarse granularity of what are in reality three-dimensional regions, and those which are 'genuinely' lower-dimensional, such as surfaces and boundaries, and which at least under certain conceptions retain this character however fine the granularity. It may be that the most appropriate theories for modelling these two kinds of lower-dimensional entity are different; the theory presented in this paper is designed to be applicable to the latter type, and it remains to be seen whether it needs to be modified in order to accommodate the former type as well.

In this paper, as in the mereotopological literature generally, no attempt has been made to distinguish between regions and the objects that can occupy them. For the purposes of representing common-sense knowledge about the world we inhabit, this distinction is crucial. Consider, for example, the notion of a *surface*.[10]. So long as we are thinking in purely geometrical terms, about regions of space, it seems reasonable to model the surface of a three-dimensional region by means of its topological boundary. If the region is a regular closed set, then its boundary may be thought of as that part of the region which is in direct contact with the outside world (i.e., the region's complement). This is a two-dimensional entity, possessing area but not volume. Now consider a physical object, for example a block of wood. At a given time, this block occupies a particular block-shaped region of space, and one might be tempted to define the surface of the block as that part of the block which occupies the surface of that region. But does any *part* of the block occupy a strictly two-dimensional region? The block has physical substance, being made of wood; any part of the block may therefore be supposed to be made of wood also. But no quantity of wood can occupy the volumeless region of space picked out by the geometrical surface of the region

[10]I am indebted to Pat Hayes for helping me to clarify my ideas on surfaces in a sequence of email exchanges in December 2003. Although what I say here has been strongly influenced by his remarks, he cannot be held responsible for any blunders, philosophical or otherwise, that I may be committing here.

occupied by the block. Following this line of thought, one might be tempted to assert that the block's surface is neither wooden nor part of the block.[11] It then becomes hard to see how the wooden surface can have the physical properties which we routinely ascribe to such surfaces: we can see it, feel it, scratch it, paint it, polish it, and so on. When we do these things, we see, feel, scratch, paint, or polish *wood*, not some immaterial mathematical abstraction. And yet there is no specifiable fraction of the wood in the block that we can single out as the wood at the surface—for example, it would be impossible to say what percentage of the wood of the block constitutes the wooden surface. All we can say is that the surface consists of the wood in the block that is available for seeing, feeling, scratching, painting, polishing, etc. Thus there is something seemingly paradoxical about physical surfaces: they seem to be made of material, without it being possible to specify precisely the material they are made of.[12]

Of course, we know that the macroscopic properties of physical lumps and their surfaces ultimately derive from the molecular constitution of the matter of which they are made, it being part of scientific physics to elaborate the detailed manner in which this happens. But this level of analysis is somewhat alien to our field of knowledge representation, where if we are concerned with physics it is primarily with *naïve* physics (Hayes 1979). Our aim is to give an account of the phenomena of the world at the level of a rational human-scale agent without specialised scientific knowledge. There is no guarantee, of course, that any such account can be both complete and consistent, and it may be that our everyday conception of matter and material objects is ultimately incoherent—indeed, one might argue that it *must* be incoherent, for otherwise we would never have been led to develop scientific physics. Substantial parts of it must be coherent, however, or we simply would not be able to work on the basis of such an conception.

Can multidimensional mereotopology, as expounded in this paper, have anything useful to say to the practitioner of knowledge representation? Our theorem establishes that a mathematically coherent account can be given of lower-dimensional spatial regions based on the notion of regular closed sets. On this picture, lower-dimensional regions are parts of the topological boundaries of regions of the next dimension up. They are regular closed sets in the subspace topologies induced on those boundaries, and the theorem establishes that the union of two such sets is again such a set. So long as we interpret all this as referring to regions of space, it seems to provide a satisfactory mathematical model. When we turn to physical objects, we noted that the true nature of physical surfaces is in some respects problematic; none the less, it would seem that there must be some relation between the physical surface of a piece of matter, and the geometrical surface of the region of space occupied by that matter. Even though we cannot simply identify the physical surface with the geometrical surface, the latter does at least constrain the position of the former.

In the case of the other class of lower-dimensional entities, e.g., two-dimensional films and membranes, or one-dimensional threads and filaments, our mathematical picture seems at best to provide a highly idealised account. Although a film or membrane might resemble a surface in being two-dimensional, there is no three-dimensional entity which it is the surface of. Moreover, its two-dimensionality is a relative matter, being dependent on the scale of resolution at which it is viewed. We know that even the most diaphanous film of material, if viewed at a sufficient magnification, will display non-negligible thickness. What makes it two-dimensional from our point of view is the ensemble of properties resulting from the fact that its thickness is orders of magnitude less than its extension in the other dimensions, enabling it, for example to be folded, wrapped around other objects, torn, etc. Similar remarks apply to one-dimensional objects such as a piece of string.

Can our mathematical characterisation of lower-dimensional regions provide support for representing this kind of entity—for example by allowing one to specify its position? That a piece of string is regarded as one-dimensional means that it can at least for some purposes be idealised to a strictly one-dimensional entity, occupying a region that may be identified with a member of one of our sets $\overline{\mathcal{P}}_1^n$. It is important for this that an element $R \in \overline{\mathcal{P}}_1^n$ may be considered in isolation from any particular $B \in \overline{\mathcal{P}}_2^n$ and $C \in \overline{\mathcal{P}}_n^n$ used to define it (as $R = C \sqcap \partial B$)—for if a one dimensional region is tied too closely to a two-dimensional region whose boundary it forms part of, then it would seem less plausible to use it to model an entity such as a piece of string which is in no way dependent on any two-dimensional entity whose surface it might inhabit. That this is perhaps reasonable follows from the fact that, although in order for R to be an element of $\overline{\mathcal{P}}_1^n$ there must *be* regions B and C such that $R = C \sqcap \partial B$, it will always be the case that there are infinitely many different possible choices of suitable regions B and C, and there is no reason to associate R more closely with one choice than with any of the others. Nonetheless, it remains true that this way of conceiving of lower-dimensional regions does not seem to sit very easily with the notion of free-standing lower-dimensional entities of the kind we have been considering.

To conclude, it is evident that while the discipline of Knowledge Representation will always have need of technical results of the kind presented in this paper, the implications of such results for the practical concerns of the field are seldom clear-cut. This is especially the case when the mathematical underpinnings of the result derive from a field (in this case, point-set topology) which was developed for quite different purposes. I thus end on a somewhat ambivalent note: on the one hand, *if* we are to apply established mathematical theories such as set theory and topology to the analysis of commonesense knowledge of the physical world, then

[11] In the terminology of (Stroll 1988), this would be to pass from a 'P-surface' (which embodies the notion that surfaces are physical entities or physical parts of physical entities) to an 'A-Surface' (which embodies the notion that surfaces are abstractions).

[12] I am here not thinking about thin films of material which may be draped over or bonded to the surface of an object, e.g., a layer of paint or varnish—think rather of an uncoated block of some homogeneous substance.

such applications should be informed by mathematical work of sufficient exactness and rigour; but on the other hand, it is unclear how far such mathematical theories are truly appropriate for the tasks in hand. The challenge to develop mathematical tools appropriate to the needs of the Knowledge Representation community is ongoing; the present paper has proposed a way of handling entities of positive codimension within a topological framework provided by regular closed sets, but as the discussion in this section suggests, the technical work needs to be supplemented by a detailed investigation into how lower-dimensional entities feature in our commonsense understanding of the world. Only then will it be possible to fully evaluate the contribution made by papers such as this one.

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