An Investigation of "Non-intermingling" Principles in Temporal Logic

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Abstract

In formalising temporal reasoning, should a proposition be allowed to change its truth value infinitely often over a finite period of time? It has been widely felt that for many types of proposition this phenomenon, which following Hamblin we call "intermingling", should be outlawed. In this paper we systematically examine the varieties of intermingling, distinguishing intermingling over an interval from intermingling at an instant. We survey the axioms that have been proposed in the literature to rule out intermingling, and determine precisely which varieties of intermingling are ruled out by which axioms. We distinguish "weak" non-intermingling principles, which only rule out one or two forms of intermingling, from "strong" non-intermingling principles which rule out all or most forms of intermingling. We show that none of the first-order solutions in the literature suffices to rule out all forms of intermingling, though a second-order solution can do so.

1 Introduction

Anyone who sets about trying to systematize the principles of temporal reasoning sooner or later comes face to face with the problem of what to do about the possibility of a proposition changing its truth value infinitely often in a finite period of time. Should one allow this to happen, and if not, how should one prevent it? This phenomenon has been given a number of different names in the literature, e.g.

"fuzz"	Prior [8]
"indefinite intermingling"	Hamblin [5]
"clustered variation"	Davis [4]

The phenomenon seems to be unavoidable; sometimes, at any rate it is harmless, and it would be pleasant if all the unavoidable instances were of the harmless kind. An example is the proposition

The temperature in Cleveland is a rational number

of which McDermott [7] says that 'assuming the temperature is smoothly changing, it will change truth value infinitely often in any finite interval'.

Another, ancient example, is provided by one of Zeno's paradoxes, the "Achilles": Zeno claimed that Achilles cannot run from A to B since first he must reach the halfway point A_1, then the three-quarters point A_2, then the seven-eighths point A_3, and so on, there being an infinite number of such divisions to traverse before reaching B. Zeno, apparently, used this as one of a set of arguments to prove that motion was impossible (so that what appears to us as motion must be some kind of illusion)—but equally we can use our conviction that motion *is* possible to show that sometimes we must allow a proposition to change truth value infinitely often in a finite interval, a suitable proposition in this case being 'Achilles has passed an even number of the divisions A_1, A_2, A_3, etc.'.

In the discussion of Zeno, Thompson [9] considered the case of a lamp which is switched alternatively on and off: on at one minute to midnight, say, then off half a minute later, then on again a quarter of a minute after that, and so on, so that the lamp is switched on and off infinitely many times during the minute leading up to midnight. This particular "supertask" is one which it will generally be agreed can under no circumstances be carried out; the proposition 'The lamp is on' can only change truth-value finitely often over a finite interval.

So there seems to be a distinction between propositions (such as 'The lamp is on') such that a change in their truth value corresponds to the kind of change (a "real" as opposed to merely conceptual change) that can only occur finitely often in a finite time, and propositions (such as 'The temperature is rational' or 'Achilles has passed an even number of divisions') which can occur infinitely often in a finite time.

It is easy to convince oneself that for practical matters of temporal reasoning, one might do well to confine one's attention to propositions of the former kind. And indeed, special names have been invented for these, e.g.,

"phenomenal predicate"	Hamblin [5]
"fact"	McDermott [7]

(Thus McDermott says that the temperature in Cleveland's being a rational number is *not* a fact.)

Having named the phenomenon one wants to outlaw, and the class of propositions for which one wants to outlaw it, one then has the task of setting up the technical apparatus to do the job. This has proved to be less easy than one might imagine. A number of different principles have been proposed for outlawing the phenomenon, and a number of different descriptions of the phenomenon have been given. As far as I know, no-one has ever examined these in a systematic way to determine precisely what each proposed principle achieves and how it is related to the others. That is the purpose of the present paper. (Note though that Davis [4] examines in some detail the implications of *allowing* certain forms of the phenomenon that others have tried to rule out.)

Before proceeding any further, we must establish at the outset the existence of two distinct phenomena to which the names 'fuzz', 'intermingling', or 'clustered variation' could apply. These can be introduced by means of two quotations from Arthur Prior, who speaks of

> ...a period of fuzziness during which between any pair of moments in which p is true there will be one at which it is false ..., and between any

> pair of moments at which p is false there will be one at which it is true[1]

and a little later in the same book

> ...what I have called a 'fuzz' of p's and Np's in the immediate past, i.e., between any past moment of p's truth and the present, however close, there is a moment of p's falsehood, and conversely [2]

adding that this construction is not available in discrete time.

So we have here two distinct phenomena, namely:

1. A proposition changing truth value infinitely often in any subinterval of a given interval. We shall call this 'intermingling over an interval'.
2. A proposition changing infinitely often in any interval ending at a given instant. We shall call this 'intermingling before an instant'.

We could add to this a "mirror-image" version of the second:

3. A proposition changing infinitely often in any interval beginning at a given instant. We shall call this 'intermingling after an instant'.

Later on we shall characterise these varieties more formally.

2 The temporal theory

In order to present a formal treatment of intermingling phenomena we must first establish the formal temporal theory we shall be working in. Some of the authors who have discussed intermingling have done so in a framework based on time instants, others have done so in a framework based on time intervals. We do not wish to prejudge this issue and shall therefore present a temporal framework that can be adapted to either possibility. Specifically, we shall introduce an instant-based theory and an interval-based theory, and show how the interval-based theory can be simulated within the instant-based theory. The general ideas are quite familiar from the literature, so the main task to be accomplished here is to establish the particular notations and axioms that we shall be using.

The interval-based theory, $\mathcal{I}$, takes intervals as the primitive units, with a primitive relation '$|$' (read 'meets'). It thus posits an ordering $(I, |)$ satisfying the axioms

I1.	$\forall i,j,k,l(i\|k \wedge i\|l \wedge j\|k \rightarrow j\|l)$
I2.	$\forall i,j,k,l(i\|k \wedge j\|l \rightarrow i\|l \oplus (i<l) \oplus (j<k))$
I3.	$\forall i \exists j,k(j\|i\|k)$
I4.	$\forall i,j,k,l(i\|j\|l \wedge i\|k\|l \rightarrow j=k)$
I5.	$\forall i,j,k,l(i\|j\|k\|l \rightarrow i<l)$
I6.	$\forall i,j,k(i\|j\|k \rightarrow \exists j',j''(i\|j'\|j''\|k))$

[1] Prior, [8], p. 94.
[2] ibid., p.108.

where '$\oplus$' denotes exclusive 'or', '$i|j|k$' abbreviates '$i|j \wedge j|k$', and '$i < j$' abbreviates '$\exists k(i|k|j)$'. Axioms I1–I5 are essentially the same, apart from differences in notation, as the axioms M1–M5 of Allen and Hayes [3]. The intuition motivating the use of '$|$' is the idea of the end of one interval being simultaneous with the beginning of the other—though these 'ends' and 'beginnings', being instants, find no expression in the formalism. Note that it can be proved from the axioms that '$|$' is irreflexive, asymmetric and antitransitive.

We shall define additional notations in $\mathcal{I}$ as follows:

'Weak precedence'	$i \lhd j =_{\mathrm{def}} (i < j) \vee i\|j$
'Subinterval'	$j \sqsubseteq i =_{\mathrm{def}} \forall k((k \lhd i \rightarrow k \lhd j) \wedge (i \lhd k \rightarrow j \lhd k))$
'Proper subinterval'	$i \sqsubset j =_{\mathrm{def}} i \sqsubseteq j \wedge i \neq j$
'Concatenation'	$i \sqcup j$ exists iff $i\|j$, and is then the unique interval such that $\forall k((k\|i \sqcup j \leftrightarrow k\|i) \wedge (i \sqcup j\|k \leftrightarrow j\|k))$

The existence of $i \sqcup j$ if $i|j$ is guaranteed by Axioms I3 and I5, while its uniqueness is guaranteed by Axiom I4.

The instant-based theory, which we shall call $\mathcal{T}$, takes instants as the primitive units of time. We posit an unbounded, dense, total ordering $(T, <)$, satisfying the axioms:

T1.	$\forall t \neg(t < t)$	(irreflexive)
T2.	$\forall t, u, v(t < u < v \rightarrow t < v)$	(transitive)
T3.	$\forall t, v \exists u(t < v \rightarrow t < u < v)$	(dense)
T4.	$\forall u \exists t, v(t < u < v)$	(unbounded)
T5.	$\forall t, u(t \leq u \vee u < t)$	(linear)

Here $t \leq u$ abbreviates $(t < u) \vee (t = u)$, and $t < u < v$ abbreviates $(t < u) \wedge (u < v)$. Two obvious models for $\mathcal{T}$, which we shall have frequent cause to refer to, are *real-numbers time*, or $\mathbb{R}$-time, in which $(T, <)$ is isomorphic to the normal ordering $(\mathbb{R}, <)$ on the real numbers, and *rational-numbers time*, or $\mathbb{Q}$-time, in which $(T, <)$ is isomorphic to the normal ordering $(\mathbb{Q}, <)$ on the rational numbers.

We can introduce intervals into $\mathcal{T}$ as follows. An interval is defined in $\mathcal{T}$ to be an ordered pair of instants (t, u), where $t < u$. (The intuition here is that an interval is a stretch of time *between* two instants; there is no presumption that an interval be identified with a *set* of instants, though this does provide a possible model for intervals if desired.) Thus

$$I = \{(t, u) \in T \times T \mid t < u\}.$$

The relation '$|$' is defined in $\mathcal{T}$ by the rule

$$(t, u)|(v, w) =_{\mathrm{def}} u = v.$$

If $i = (t, u)$ then we put $\mathit{inf}(i) \equiv t, \mathit{sup}(i) \equiv u$. Thus the definition of '$|$' could equivalently be given as

$$i|j =_{\mathrm{def}} \mathit{sup}(i) = \mathit{inf}(j).$$

We shall say that instant t *limits* interval i, written $\mathit{Lim}(t, i)$, if $t = \mathit{inf}(u)$ or $t = \mathit{sup}(u)$, and that t *falls within* i, written $t \sqsubset\!\!\!\!= i$, if $\mathit{inf}(i) < t < \mathit{sup}(i)$. Our defined notations for intervals can then be characterised in $\mathcal{T}$ as follows:

$j \sqsubseteq i \Leftrightarrow \forall t(t \in j \rightarrow t \in i)$.
$i \sqcup j$ exists iff $i|j$, and then $i \sqcup j = (inf(i), sup(j))$.
$i < j \Leftrightarrow sup(i) < inf(j)$

Note the 'polymorphic' character of '<' as a relation either between instants or between intervals; it could also relate an instant and an interval, in either order:

$$t < i \Leftrightarrow t < inf(i), \quad i < t \Leftrightarrow sup(i) < t.$$

It would be pedantic to insist on separate symbols for the four possible cases; and besides, there is the same underlying intuition for all four of them.

3 Introducing fluents

A theory of time alone is insufficient to generate change; we need to have states of affairs holding or not holding at different times. The fundamental notion is that of a *fluent*, which is a function on times. This conforms to the eighteenth-century usage of Newton rather than the usage popularised by McCarthy and Hayes [6], whereby a fluent is a function on *situations* rather than times. We shall restrict our attention to *Boolean* fluents, i.e., functions from times to truth-values. Our 'times' will be instants or intervals depending on the system we are working in. For a Boolean fluent ϕ, we shall write

$At(\phi, t)$ to say that ϕ holds at instant t
$On(\phi, i)$ to say that ϕ holds throughout interval i.

(So $On(\phi, i)$ is the same as the $Holds(\phi, i)$ of [1].) We shall use At as a primitive predicate in $\mathcal{T}$ and On as a primitive predicate in $\mathcal{I}$. In $\mathcal{T}$, we shall define On in terms of At. We shall also make frequent use of the notion of the *negation* of a fluent: the negation of ϕ will be written $-\phi$.

In the instant-based system $\mathcal{T}$, we need only two axioms, as follows:

INT $\forall i(On(\phi, i) \leftrightarrow \forall t \in i At(\phi, t))$
NEG $\forall t(At(-\phi, t) \leftrightarrow \neg At(\phi, t))$

These axioms serve in effect as definitions of the notations $On(\phi, i)$ and $-\phi$ in $\mathcal{T}$.

Note that, given **INT** and **NEG**, we have

$$At(-(-\phi), t) \Leftrightarrow \neg At(-\phi, t) \Leftrightarrow \neg\neg At(\phi, t) \Leftrightarrow At(\phi, t).$$

so $-(-\phi)$ can be regarded as the same fluent as ϕ. The equivalence extends to holding over an interval as well, for

$$On(-(-\phi), i) \Leftrightarrow \forall t \in i At(-(-\phi), t) \Leftrightarrow \forall t \in i At(\phi, t) \Leftrightarrow On(\phi, i)$$

In the interval system $\mathcal{I}$, neither **INT** nor **NEG** makes sense, since they involve reference to instants. Instead, we make use of the following axioms

HOM $\forall i(On(\phi, i) \rightarrow \forall j \sqsubseteq i On(\phi, j))$
NEG* $\forall i(On(-\phi, i) \leftrightarrow \forall j \sqsubseteq i \neg On(\phi, j))$.
Ax10 $\forall i[(\forall j \sqsubseteq i \exists k \sqsubseteq j On(\phi, k)) \rightarrow On(\phi, i)]$.

Historically, **HOM** can be recognised as Axiom 9 of [5] and H1 of [1]. Our label reflects the idea that the state-like fluents we are dealing with occupy intervals *homogeneously*. **NEG*** is given by Hamblin as the definition of negation, while for Allen it is axiom H4. The converse of **Ax10** is implied by **HOM**. Our label for this formula refers to the fact that it is Hamblin's Axiom 10; Allen's axiom H2 is just **Ax10** conjoined with its converse.

Note that in $\mathcal{I}$ we still have the equivalence of $-(-\phi)$ and ϕ, since we have

$$\begin{array}{rcll} On(-(-\phi), i) & \Leftrightarrow & \forall j \sqsubseteq i \neg On(-\phi, j) & (\text{By } \mathbf{NEG^*}) \\ & \Leftrightarrow & \forall j \sqsubseteq i \neg \forall k \sqsubseteq j \neg On(\phi, k) & (\text{By } \mathbf{NEG^*}) \\ & \Leftrightarrow & \forall j \sqsubseteq i \exists k \sqsubseteq j On(\phi, k) & \\ & \Leftrightarrow & On(\phi, i) & (\text{By } \mathbf{Ax10} \text{ and its converse}) \end{array}$$

This provides, incidentally, a clear motivation for including the otherwise rather unobvious **Ax10** as an axiom.

Given that formulae of the form $On(\phi, i)$ exist in both $\mathcal{T}$ and $\mathcal{I}$, it is natural to ask whether we can prove the axioms of $\mathcal{I}$ in $\mathcal{T}$. It turns out that we cannot do so unless we add an extra axiom to $\mathcal{T}$; if we do this, the resulting system perfectly simulates $\mathcal{I}$ within an instants-only framework; we shall call it $\mathcal{T}^*$.

We begin by investigating how far $\mathcal{I}$ can be simulated within $\mathcal{T}$ without the addition of extra axioms.

First, we can show that **HOM** is a consequence of **INT**: For, given **INT**, we have

$$\begin{array}{rcl} On(\phi, i) \wedge j \sqsubseteq i & \Rightarrow & \forall t \sqsubseteq i At(\phi, t) \wedge j \sqsubseteq i \\ & \Rightarrow & \forall t \sqsubseteq j At(\phi, t) \\ & \Rightarrow & On(\phi, j) \end{array}$$

Note that although **INT** $\Rightarrow$ **HOM**, the converse does not hold: **HOM** by itself says nothing about how truth values should be assigned at instants.

Second, one direction of the biconditional in **NEG*** follows from **NEG** and **INT**:

$$\begin{array}{rcl} j \sqsubseteq i \wedge On(-\phi, i) & \Leftrightarrow & j \sqsubseteq i \wedge \forall t \sqsubseteq i At(-\phi, t) \\ & \Rightarrow & \forall t \sqsubseteq j At(-\phi, t) \\ & \Leftrightarrow & \forall t \sqsubseteq j \neg At(\phi, t) \\ & \Rightarrow & \neg \forall t \sqsubseteq j At(\phi, t) \\ & \Leftrightarrow & \neg On(\phi, j) \end{array}$$

The converse does not follow, though it would appear that any counterexample must involve intermingling of a kind that cannot occur in $\mathcal{I}$. For an example in $\mathbb{R}$-time, let ϕ be true at rational instants and false at irrational ones, within some interval i. Then we have $\forall j \sqsubseteq i \neg On(\phi, j)$ but we do not have $On(-\phi, i)$.

Third, **HOM** and **NEG** do not imply **Ax10**. To see this, a simple counterexample suffices. Assign truth values to ϕ by the rule $At(\phi, t) \leftrightarrow t \neq t_0$. Then since every interval has a subinterval not containing t_0, the antecedent of **Ax10** holds for any interval i. But if i contains t_0, the consequent of **Ax10** does not hold.

It thus appears that in order to simulate $\mathcal{I}$ within $\mathcal{T}$ we need to add extra axioms strong enough to give us **Ax10** and the right-to-left implication in **NEG***.

It turns out that there is a single very well-motivated axiom that will do the trick. The interval system $\mathcal{I}$ differs from the instant system $\mathcal{T}$ in that it attaches no independent meaning to the notion of a fluent holding at an instant; when I say 'no independent meaning' I have in mind the idea that, for instance, if ϕ holds on interval i, and $t \sqsubseteq i$, then even though it is natural to say that ϕ holds at t, it only does so by virtue of holding over an interval within which t falls. One possibility might therefore be to say that this is the *only* condition under which it makes sense to say that a fluent holds at an instant, in other words to posit

$$\forall t(At(\phi, t) \leftrightarrow \exists i(t \sqsubseteq i \wedge On(\phi, i))).$$

This would, however lead to there being instants at which neither ϕ nor $-\phi$ holds, thus contradicting **NEG**. This would happen for any instant which is the meeting point of two intervals, over one of which ϕ, and over the other $-\phi$, holds. When this happens, we must assume that one or other of ϕ, $-\phi$ holds, and for the purposes of simulating $\mathcal{I}$ in $\mathcal{T}$ it des not matter which. On the other hand, if t is the meeting point of two intervals over both of which ϕ holds, then we must insist that ϕ holds at t as well. In any case, therefore, ϕ can only hold at an instant t if it holds over an interval limited by t, giving us our required axiom

INST $\quad \forall t(At(\phi, t) \rightarrow \exists i(Lim(t, i) \wedge On(\phi, i)))$

We shall show that **INST** implies both **Ax10** and the right-to-left implication in **NEG***.

By way of preparation, we introduce a simple lemma

Lemma1 $\quad \exists t, j(t \sqsubseteq i \wedge Lim(t, j) \wedge On(\phi, j)) \rightarrow \exists k \sqsubseteq i On(\phi, k)$

Proof Assume $t \sqsubseteq i$, $Lim(t, j)$, and $On(\phi, j)$. Then either $t = inf(j)$ or $t = sup(j)$. If $t = inf(j)$, then we can put $j = (t, t')$. There are two cases to consider:

- If $t' \leq sup(i)$ then $j \sqsubseteq i$ so we have $j \sqsubseteq i \wedge On(\phi, j)$, satisfying the consequent of the lemma (with $k = j$).
- If $sup(i) < t'$ then $(t, sup(i)) \sqsubseteq i$ and $(t, sup(i)) \sqsubseteq j$. Since we have $On(\phi, j)$, the latter implies $On(\phi, (t, sup(i)))$, by **HOM**. Hence we have $(t, sup(i)) \sqsubseteq i \wedge On(\phi, (t, sup(i)))$, again satisfying the consequent of the lemma (with $k = (t, sup(i))$).

The case $t = sup(j)$ is handled in a precisely analogous way.

We now turn to the proofs of **Ax10** and part of **NEG***.

First, assuming **INST** we have

$$\begin{array}{lcll}
\forall j \sqsubseteq i \exists k \sqsubseteq j On(\phi, k) & \Rightarrow & \forall j \sqsubseteq i \exists k \sqsubseteq j \neg On(-\phi, k) & \\
 & \Leftrightarrow & \forall j \sqsubseteq i \neg \forall k \sqsubseteq j On(-\phi, k) & \\
 & \Rightarrow & \forall j \sqsubseteq i \neg On(-\phi, j) & \text{(By } \mathbf{HOM}) \\
 & \Leftrightarrow & \neg \exists j \sqsubseteq i On(-\phi, j) & \\
 & \Rightarrow & \neg \exists j \exists t \sqsubseteq i(Lim(t, j) \wedge On(-\phi, j)) & \text{(By } \mathbf{Lemma1}) \\
 & \Leftrightarrow & \forall t \sqsubseteq i \neg \exists j(Lim(t, j) \wedge On(-\phi, j)) & \\
 & \Rightarrow & \forall t \sqsubseteq i \neg At(-\phi, t) & \text{(By } \mathbf{INST}) \\
 & \Leftrightarrow & \forall t \sqsubseteq i At(\phi, t) & \text{(By } \mathbf{NEG}) \\
 & \Leftrightarrow & On(\phi, i) & \text{(By } \mathbf{INT})
\end{array}$$

and hence we have $\mathcal{T}$+**INST**$\Rightarrow$**Ax10**.

Second, again assuming **INST**, we have

$$\begin{array}{lll}
\forall j \sqsubseteq i \neg On(\phi, j) & \Leftrightarrow \ \forall j \sqsubseteq i \neg \forall t \Subset j At(\phi, t) & \text{(By } \mathbf{INT}) \\
& \Leftrightarrow \ \forall j \sqsubseteq i \exists t \Subset j \neg At(\phi, t) & \\
& \Leftrightarrow \ \forall j \sqsubseteq i \exists t \Subset j At(-\phi, t) & \text{(By } \mathbf{NEG}) \\
& \Rightarrow \ \forall j \sqsubseteq i \exists t \Subset j \exists k (Lim(t, k) \wedge On(-\phi, k)) & \text{(By } \mathbf{INST}) \\
& \Rightarrow \ \forall j \sqsubseteq i \exists k \sqsubseteq j On(-\phi, k) & \text{(By } \mathbf{Lemma1}) \\
& \Rightarrow \ On(-\phi, i) & \text{(By } \mathbf{Ax10})
\end{array}$$

thus proving the right-to-left implication in **NEG***.

Our overall result, then, is that

$$\mathbf{INT} + \mathbf{NEG} + \mathbf{INST} \Rightarrow \mathbf{HOM} + \mathbf{Ax10} + \mathbf{NEG}^*.$$

Our system $\mathcal{T}$ has axioms **INT** and **NEG** only; $\mathcal{I}$ has axioms **HOM**, **Ax10**, and **NEG***. The latter system is simulated in the former by addition of **INST**, the system $\mathcal{T}$ + **INST** being denoted $\mathcal{T}^*$.

4 The "non-intermingling" principles

In this section we shall introduce a number of principles that have figured in the literature with the express intention of ruling out intermingling.

4.1 Hamblin and Allen

Hamblin [5] states that **Ax10** together with **NEG*** implies

(**HAM**) $$\forall i (\neg On(\phi, i) \rightarrow \exists j \sqsubseteq i (On(-\phi, j))),$$

of which he says

> This axiom excludes the possibility of indefinitely finely intermingled periods of redness and non-redness, or of high and low pitch, or of whatever else it is that we take the elementary predicates to represent. [3]

The 'discrete variation axiom schema' of [2] is exactly **HAM** (allowing for differences of notation).

Hamblin's claim is that

$$\mathbf{Ax10} + \mathbf{NEG}^* \Rightarrow \mathbf{HAM}.$$

To verify this, recall that **NEG*** and **Ax10** imply the equivalence of $-(-\phi)$ with ϕ. Using this equivalence, we can rewrite **NEG*** as

$$\forall i (\neg On(\phi, i) \leftrightarrow \neg \forall j \sqsubseteq i \neg On(-\phi, j));$$

but this is equivalent to

$$\forall i (\neg On(\phi, i) \leftrightarrow \exists j \sqsubseteq i On(-\phi, j)),$$

[3] Hamblin [5], pp.131–2.

which implies **HAM**.

Alternatively, arguing in $\mathcal{T}^*$, we can derive **HAM** directly from **INST**, as follows:

$$\begin{array}{rcll} \neg On(\phi, i) & \Leftrightarrow & \neg\forall t \sqsubseteq i At(\phi, t) & \text{(By } \mathbf{INT}) \\ & \Leftrightarrow & \exists t \sqsubseteq i \neg At(\phi, t) & \\ & \Leftrightarrow & \exists t \sqsubseteq i At(-\phi, t) & \text{(By } \mathbf{NEG}) \\ & \Rightarrow & \exists t \sqsubseteq i \exists j(Lim(t, j) \wedge On(-\phi, j)) & \text{(By } \mathbf{INST}) \\ & \Rightarrow & \exists j \sqsubseteq i On(-\phi, j) & \text{(By } \mathbf{Lemma1}) \end{array}$$

Note that although **INST** implies **HAM**, the converse does not hold. A counterexample involves intermingling on both sides of a instant: assume we have

$$t_1 < t_2 < t_3 < t_4 < \cdots < u < \cdots v_4 < v_3 < v_2 < v_1$$

where u is at once the earliest instant later than all the t_r and the latest instant earlier than all the v_s, and assign truth values to ϕ by the rule that ϕ holds at t if and only if one of the following holds:

- $t = u$
- For some n, $t_{2n} \leq t < t_{2n+1}$
- For some n, $v_{2n+1} < t \leq v_{2n}$

Then **HAM** is satisfied, since any interval over which ϕ does not hold has a subinterval over which $-\phi$ holds, and conversely any interval over which $-\phi$ fails to hold has a subinterval over which ϕ holds; but **INST** is not satisfied since we have

$$At(\phi, u) \wedge \forall i(Lim(u, i) \rightarrow \neg On(\phi, i)).$$

Note also that **Ax10** without **NEG*** does not imply **HAM**. A counterexample involves intermingling over an interval; suppose that i has *no* subinterval over which ϕ is constant in truth-value. Then **Ax10** holds for the uninteresting reason that its antecedent is false for every interval i, since there are *no* intervals $k \sqsubseteq i$ such that $On(\phi, k)$. On the other hand **HAM** does not hold for this set-up, since we have $\forall j \sqsubseteq i \neg On(\phi, j)$ without having $On(-\phi, j)$.

4.2 McDermott and Davis

McDermott [7] developed a temporal theory for AI; the formulae are presented in a Lisp-like notation that has not found favour in the temporal reasoning community, and hence looks pretty unreadable to us nowadays. The relevant formula, for our purposes, is his Axiom 9, formulated by Ernest Davis; shorn of its reference to variant possible futures (which is irrelevant for our present purposes) this axiom could be translated into our notation as:

(DAV) $$\forall\phi\forall t(\exists u Constant(\phi, (u, t)) \wedge \exists u Constant(\phi, (t, u)))$$

where $Constant(\phi, i)$ is an abbreviation for $On(\phi, i) \vee On(-\phi, i)$.

We can split **DAV** up into two formulae

(DAVL) $$\forall t \exists u Constant(\phi, (u, t))$$

(**DAVR**) $\forall t \exists u Constant(\phi, (t, u))$

These can be given in a pure interval formulation as follows:

(**DAVL***) $\forall i \exists j (j|i \wedge Constant(\phi, j))$

(**DAVR***) $\forall i \exists j (i|j \wedge Constant(\phi, j))$

That these are equivalent to the formulations in terms of instants can be shown as follows.

First, assume **DAVR**, and let $i = (s, t)$ be any interval. Then by **DAVR**,

$$\exists u Constant(\phi, (t, u)).$$

But $i|(t, u)$, so

$$\exists j (i|j \wedge Constant(\phi, j)).$$

Hence **DAVR** $\Rightarrow$ **DAVR***.

Conversely, assume **DAVR***, and let t be any instant. Choose any $s < t$; then by **DAVR***,

$$\exists j ((s, t)|j \wedge Constant(\phi, j)).$$

Let $j = (t, u)$. Then

$$\exists u (t < u \wedge Constant(\phi, (t, u))).$$

Hence **DAVR*** $\Rightarrow$ **DAVR**.

Exactly analogous arguments apply to **DAVL** and **DAVL***

4.3 The Finite Decomposition Property

Davis [4] gives an axiom equivalent to **DAV** and notes that in $\mathbb{R}$-time this is equivalent to the property that any bounded interval contains only finitely many maximal subintervals over which ϕ has constant truth-value. We shall call this the *finite decomposition property*, and write it as

FD For every ϕ and every interval i there is a decomposition

$$i = i_1 \sqcup i_2 \sqcup \cdots \sqcup i_n$$

such that for $r = 1, \ldots, n$, $Constant(\phi, i_r)$.

As Davis points out, this is a second-order property, since to state it we have to quantify over sets of intervals.

4.4 The Transition Point property

Allen and Ferguson [2] state a 'theorem useful for reasoning about when properties change truth values', which in our notation can be written as

(**TP**) $\forall i, j (i \lhd j \wedge On(\phi, i) \wedge On(-\phi, j) \rightarrow$
$\exists k, l (k|l \wedge j \sqsubseteq l \wedge On(\phi, k) \wedge On(-\phi, l)),$

Here **TP** stands for 'transition point', the motivation being:

> Defining a *transition point* to be the existence of adjacent intervals for which a property has complementary truth values, this axiom states that if a property has changed truth value, then there must be a transition point where it does so.[4]

Allen and Ferguson claim that this follows from **HAM** and **HOM**, but as we shall see below, this is incorrect.

5 Defining intermingling

Our purpose is to investigate the relationships between the various non-intermingling principles discussed in the previous section and to determine precisely what kinds of intermingling they succeed in ruling out. To prepare for this we must first discuss intermingling itself in more detail.

As we have seen, Prior's 'fuzz' comes in two varieties, which we have called intermingling over an interval and intermingling in the neighbourhood of an instant. The latter in turn has three subvarieties, namely intermingling immediately before an instant, intermingling immediately after an instant, and intermingling on both sides of an instant. We give below exact characterisations of these four forms of intermingling; by a 'ϕ-interval' we mean an interval i such that $On(\phi, i)$, and by a 'ϕ-instant' we mean an instant t such that $At(\phi, t)$.

- *Intermingling with respect to ϕ over an interval i* occurs if between any two ϕ-instants (respectively $(-\phi)$-instants) in i there is a $(-\phi)$-instant (respectively ϕ-instant). This is expressed by the predicate *Intfuzz*, where

$$\begin{aligned} \mathit{Intfuzz}(\phi, i) \quad =_{\mathrm{def}} \quad & \forall (t, t') \sqsubseteq i \exists t'' \in (t, t')[\\ & (At(\phi, t) \wedge At(\phi, t') \rightarrow At(-\phi, t'')) \ \wedge \\ & (At(-\phi, t) \wedge At(-\phi, t') \rightarrow At(\phi, t''))] \end{aligned}$$

  Note that *Intfuzz* is symmetric as between ϕ and $-\phi$, i.e.,

$$\forall \phi, i(\mathit{Intfuzz}(\phi, i) \leftrightarrow \mathit{Intfuzz}(-\phi, i)).$$

- *Intermingling with respect to ϕ immediately before an instant t* (for short 'prefuzz on ϕ at t') occurs if between any ϕ-instant (respectively $(-\phi)$-instant) earlier than t and t itself there is a $(-\phi)$-instant (respectively ϕ-instant). This is expressed by

$$\mathit{Prefuzz}(\phi, t) =_{\mathrm{def}} \forall t' < t \exists t'' \in (t', t)(At(\phi, t') \leftrightarrow At(-\phi, t'')).$$

  Again, we have symmetry:

$$\forall \phi, t(\mathit{Prefuzz}(\phi, t) \leftrightarrow \mathit{Prefuzz}(-\phi, t)).$$

[4] Allen and Ferguson [2], p.541.

- *Intermingling with respect to ϕ immediately after an instant t* (for short 'postfuzz on ϕ at t') occurs if between t and any ϕ-instant (respectively $(-\phi)$-instant) later than t there is a $(-\phi)$-instant (respectively ϕ-instant). This is expressed by
$$Postfuzz(\phi, t) =_{\text{def}} \forall t' > t \exists t'' \in (t, t')(At(\phi, t') \leftrightarrow At(-\phi, t'')).$$

Note that both *Prefuzz* and *Postfuzz* can be given pure interval variants as follows:

$$\begin{array}{rcl} Prefuzz^*(\phi, i) & =_{\text{def}} & \forall j < i \exists k(j \lhd k \lhd i \wedge (On(\phi, k) \leftrightarrow On(-\phi, j)) \\ Postfuzz^*(\phi, i) & =_{\text{def}} & \forall j > i \exists k(i \lhd k \lhd j \wedge (On(\phi, k) \leftrightarrow On(-\phi, j)) \end{array}$$

We must check the relationship between $Prefuzz^*$ and *Prefuzz*, and likewise with the *Postfuzz* rules.

Assume $Prefuzz(\phi, inf(i))$ and $j < i$. Suppose first that $On(-\phi, j)$. Then either $At(\phi, sup(j))$ or $At(-\phi, sup(j))$.

If $At(\phi, sup(j))$, then by **INST** we have $\exists k(Lim(sup(j), k) \wedge On(\phi, k))$. The first conjunct $Lim(sup(j), k)$ means that either $sup(j) = sup(k)$ or $sup(j) = inf(k)$. We cannot have the former, since in that case either $j \sqsubseteq k$ or $k \sqsubseteq j$, either of which is incompatible with $On(\phi, k) \wedge On(-\phi, j)$. Hence $sup(j) = inf(k)$, and $j \lhd k$. But also $k \lhd i$, since otherwise $inf(i) \sqsubseteq k$, which given $On(\phi, k)$ is incompatible with $Prefuzz(\phi, inf(i))$. Hence $j \lhd k \lhd i \wedge On(\phi, k)$.

If on the other hand $At(-\phi, sup(j))$, then we must have $\exists t \sqsubseteq (sup(j), inf(i)) At(\phi, t)$ (from $Prefuzz(\phi, inf(i))$). By **INST**, $\exists k(Lim(t, k) \wedge On(\phi, t))$, so by **Lemma1** we have $\exists k \sqsubseteq (sup(j), inf(i)) On(\phi, k)$, i.e., $\exists k(j \lhd k \lhd i \wedge On(\phi, k))$.

Hence in either case we have

$$\exists k(j \lhd k \lhd i \wedge (On(-\phi, j) \rightarrow On(\phi, k)).$$

Suppose next that $\neg On(-\phi, j)$. By **NEG***, $\exists j' \sqsubseteq j On(\phi, j')$. We can now argue exactly as before to show that

$$\exists k(j' \lhd k \lhd i \wedge (\neg On(-\phi, j) \rightarrow On(-\phi, k)).$$

But this implies
$$\exists k(j \lhd k \lhd i \wedge (\neg On(-\phi, j) \rightarrow \neg On(\phi, k)).$$

Combining this with our earlier result we obtain

$$\exists k(j \lhd k \lhd i \wedge (On(\phi, k) \leftrightarrow On(-\phi, j))),$$

and since this applies to any $j < i$, we have $Prefuzz^*(\phi, i)$ as required. We thus have

$$\textbf{INST} \Rightarrow \forall \phi, i(Prefuzz(\phi, inf(i)) \rightarrow Prefuzz^*(\phi, i)).$$

Conversely, assume we have $Prefuzz^*(\phi, i)$, and suppose $(t' < inf(i)) \wedge At(\phi, t')$. Assuming **INST**, there is an interval j such that $Lim(t', j) \wedge On(\phi, j)$. By Lemma 1, there is j such that $(j < i) \wedge On(\phi, j)$. Then from $Prefuzz^*(\phi, i)$, there is an interval k such that $j \lhd k \lhd i \wedge On(-\phi, k)$. Choose any instant $t'' \sqsubseteq k$; then $t' < t'' < inf(i)$, and by **INT**, $At(-\phi, t'')$. We can similarly argue that if $At(-\phi, t')$ then $At(\phi, t'')$. Thus we have

$$\exists t'' \sqsubseteq (t', \mathit{inf}(i))(At(\phi, t') \leftrightarrow At(-\phi, t'')),$$

and we have verified $\mathit{Prefuzz}(\phi, \mathit{inf}(i))$, giving us

$$\mathbf{INST} \Rightarrow \forall \phi, i(\mathit{Prefuzz}^*(\phi, i) \rightarrow \mathit{Prefuzz}(\phi, \mathit{inf}(i))).$$

In sum, we have shown that

$$\mathbf{INST} \Rightarrow \forall \phi, i(\mathit{Prefuzz}^*(\phi, i) \leftrightarrow \mathit{Prefuzz}(\phi, \mathit{inf}(i))).$$

It can similarly be shown that

$$\mathbf{INST} \Rightarrow \forall \phi, i(\mathit{Postfuzz}^*(\phi, i) \leftrightarrow \mathit{Postfuzz}(\phi, \mathit{sup}(i))).$$

In intervals-only time, therefore, $\mathit{Prefuzz}^*$ and $\mathit{Postfuzz}^*$ can be used in place of $\mathit{Prefuzz}$ and $\mathit{Postfuzz}$ to characterise intermingling before and after an instant.

If we don't assume **INST**, however, the two pairs of conditions are different. To see this, suppose that in $\mathbb{R}$-time we assign truth-values to ϕ according to the rule

$$At(\phi, t) \Leftrightarrow \exists n \in \mathbb{Z}(t = 2^n).$$

Then both $\mathit{Prefuzz}(\phi, 0)$ and $\mathit{Postfuzz}(\phi, 0)$ hold, but for no x do we have either $\mathit{Prefuzz}^*(\phi, (0, x))$ or $\mathit{Postfuzz}^*(\phi, (x, 0))$. Of course this particular truth-value assignment is impossible under **INST**.

Is there an intervals-only formulation of $\mathit{Intfuzz}$? Assuming $\mathit{Intfuzz}(\phi, i)$, suppose $\exists j \sqsubseteq i On(\phi, j)$. Then let $t, t' \sqsubseteq j$, so we have $At(\phi, t) \wedge At(\phi, t')$. By $\mathit{Intfuzz}(\phi, i)$, there is an instant $t'' \sqsubseteq (t, t')$ such that $At(-\phi, t'')$. But since $(t, t') \sqsubseteq j$, we know $t'' \sqsubseteq j$, so $At(\phi, t'')$, a contradiction. Hence we have

$$\neg \exists j \sqsubseteq i On(\phi, j).$$

Similarly we can show

$$\neg \exists j \sqsubseteq i On(-\phi, j).$$

This suggests that an appropriate interval formulation of $\mathit{Intfuzz}$ would have to be

$$\mathit{Intfuzz}^*(\phi, i) =_{\text{def}} \forall j \sqsubseteq i(\neg On(\phi, j) \wedge \neg On(-\phi, j)).$$

which may be written more compactly as

$$\mathit{Intfuzz}^*(\phi, i) =_{\text{def}} \forall j \sqsubseteq i \neg Constant(\phi, j).$$

We have just shown that $\mathit{Intfuzz}(\phi, i) \Rightarrow \mathit{Intfuzz}^*(\phi, i)$; the converse holds too. For assume $\mathit{Intfuzz}^*(\phi, i)$, and let $(t, t') \sqsubseteq i$ be such that $At(\phi, t) \wedge At(\phi, t')$. By $\mathit{Intfuzz}^*(\phi, i)$, we have

$$\begin{aligned}
\neg On(\phi, (t, t')) \wedge \neg On(-\phi, (t, t')) &\Leftrightarrow \neg \forall x \sqsubseteq (t, t') At(\phi, x) \wedge \neg \forall x \sqsubseteq (t, t') At(-\phi, x) \\
&\Leftrightarrow \exists x \sqsubseteq (t, t') \neg At(\phi, x) \wedge \exists x \sqsubseteq (t, t') \neg At(-\phi, x) \\
&\Leftrightarrow \exists x \sqsubseteq (t, t') At(-\phi, x) \wedge \exists x \sqsubseteq (t, t') At(\phi, x) \\
&\Rightarrow \exists x \in (t, t') At(-\phi, x)
\end{aligned}$$

which verifies that $Intfuzz(\phi, i)$.

Suppose now that we have **INST**. For any $t \sqsubseteq i$ we have

$$At(\phi, t) \vee \neg At(\phi, t),$$

i.e., $At(\phi, t) \vee At(-\phi, t)$, so by **INST** we have

$$\exists j(Lim(t, j) \wedge (On(\phi, j) \vee On(-\phi, j))).$$

By Lemma 1, this implies

$$\exists j' \sqsubseteq i(On(\phi, j') \vee On(-\phi, j')),$$

i.e.,

$$\exists j' \sqsubseteq iConstant(\phi, j'),$$

contradicting $Intfuzz^*(\phi, i)$.

We have now shown, therefore, that

$$\begin{array}{rcl} \forall \phi, i(Intfuzz(\phi, i) & \leftrightarrow & Intfuzz^*(\phi, i)) \\ \mathbf{INST} & \Rightarrow & \forall \phi, i \neg Intfuzz^*(\phi, i) \end{array}$$

Thus, in particular, **INST** in itself functions as a non-intermingling principle, ruling out intermingling over an interval.

6 What kinds of intermingling are ruled out by the non-intermingling principles?

In this section we shall consider each of the non-intermingling principles in turn and investigate which kinds of intermingling it rules out.

It will be convenient to use a standard notation for examples of the different kinds of intermingling at an instant. For this purpose we will always assume that $t_0, t_1, t_2, \ldots$ are a sequence of instants converging on instant u from the past, while $v_0, v_1, v_2, v_3, \ldots$ are a sequence of instants converging on u from the future, so we have

$$t_0 < t_1 < t_2 < t_3 < \cdots < u < \cdots < v_3 < v_2 < v_1 < v_0.$$

(For example, we could put, in $\mathbb{R}$-time or $\mathbb{Q}$-time, $u = 0$, $t_i = -2^{-i}$, and $v_i = 2^{-i}$.) We shall also put

$$i_n = (t_n, t_{n+1}), \quad j_n = (v_{n+1}, v_n),$$

giving us intervals such that:

$$i_0|i_1|i_2|i_3| \cdots < \cdots |j_3|j_2|j_1|j_0,$$

the picture appropriate to an intervals-only formulation.

We shall make use of standard truth-assignments for a proposition ϕ as follows:

— (PREFUZZ*) For $n = 0, 1, 2, 3, \ldots$ put

$$On(\phi, i_{2n}) \wedge On(-\phi, i_{2n+1}).$$

The value of ϕ at t_n, at u, and at any time in the future of u, is left unspecified.

— (POSTFUZZ*) For $n = 0, 1, 2, 3, \ldots$ put

$$On(\phi, j_{2n}) \wedge On(-\phi, j_{2n+1}).$$

The value of ϕ at v_n, at u, and at any time in the past of u, is left unspecified.

— (PREFUZZ) For $n = 0, 1, 2, 3, \ldots$, put

$$At(\phi, t_n) \wedge On(-\phi, i_n).$$

The value of ϕ at u and at any time in the future of u is left unspecified.

— (POSTFUZZ) For $n = 0, 1, 2, 3, \ldots$, put

$$At(\phi, v_n) \wedge On(-\phi, j_n).$$

The value of ϕ at u and at any time in the past of u is left unspecified.

— (PRECONST) ϕ holds constantly before u, i.e.,

$$\forall t < u On(\phi, (t, u)).$$

— (POSTCONST) ϕ holds constantly after u, i.e.,

$$\forall v > u On(\phi, (u, v)).$$

Note that, if i and j are intervals such that $sup(i) = u = inf(j)$, then

- PREFUZZ* satisfies both $Prefuzz(\phi, u)$ and $Prefuzz^*(\phi, j)$;
- PREFUZZ satisfies $Prefuzz(\phi, u)$ but does not satisfy $Prefuzz^*(\phi, j)$.
- POSTFUZZ* satisfies both $Postfuzz(\phi, u)$ and $Postfuzz^*(\phi, i)$;
- POSTFUZZ satisfies $Postfuzz(\phi, u)$ but does not satisfy $Postfuzz^*(\phi, i)$.

We will generally specify a truth assignment to ϕ by combining one of the PRE- conditions from the above list with one of the POST- conditions.

6.1 HAM

6.1.1 HAM rules out intermingling over an interval

The principle **HAM** is expressed by the formula

$$\forall i(\neg On(\phi, i) \rightarrow \exists j \sqsubseteq i(On(-\phi, j))).$$

Since $Constant(\phi, j) \Leftrightarrow On(\phi, j) \vee On(-\phi, j)$, **HAM** implies

$$\forall i(\neg On(\phi, i) \rightarrow \exists j \sqsubseteq i Constant(\phi, j)),$$

and from the definition of $Constant$ we have

$$\forall i(On(\phi, i) \rightarrow \exists j \sqsubseteq i Constant(\phi, j)).$$

Hence **HAM** implies

$$\forall i \exists j \sqsubseteq i Constant(\phi, j)$$

i.e., $\forall i \neg Intfuzz^*(\phi, i)$. Hence **HAM** rules out intermingling over an interval.

6.1.2 HAM does not rule out intermingling at an instant

However, **HAM** does not rule out any of the other kinds of intermingling. Consider for example, the truth-assignment PREFUZZ*+POSTCONST. Now take any interval $i = (x, y)$ such that $\neg On(\phi, i)$. Clearly we must have $x < u$, otherwise i would be a subinterval of (u, y), and we would have $On(\phi, i)$. We are looking for a subinterval j of i such that $On(-\phi, j)$.

Let n be the greatest integer such that $x \leq t_{2n}$. Then if $t_{2n} < y$ we have

$$(t_{2n}, y) \sqsubseteq i_{2n}, \quad (t_{2n}, y) \sqsubseteq (x, y),$$

so we can put $j = (t_{2n}, y)$ and we have $On(-\phi, j)$. If $y \leq t_{2n}$ then we must have $x < t_{2n-1}$, for otherwise we would have $(x, y) \sqsubseteq i_{2n-1}$, giving $On(\phi, i)$ contrary to our hypothesis. If $y \leq t_{2n-1}$ then $(x, y) \sqsubseteq t_{2n-2}$, so $On(-\phi, (x, y))$ and we can put $j = (x, y)$. If on the other hand $y > t_{2n-1}$, we put $j = (x, t_{2n-1})$.

This exhausts the possibilities; so in any case we conclude that i has a subinterval j such that $On(-\phi, j)$, and we have shown that **HAM** holds for this set-up. Since both *Prefuzz*(ϕ, u) and *Prefuzz*$^*(\phi, j)$ for any $j > u$ also hold here, **HAM** does not preclude either of them.

Arguing from the temporal mirror image of this set-up gives us a similar conclusion in regards to *Postfuzz* and *Postfuzz**.

6.2 INST

The principle **INST** is expressed by the formula

$$\forall t(At(\phi, t) \rightarrow \exists i(Lim(t, i) \wedge On(\phi, i))).$$

As already shown, this implies **HAM**, and therefore rules out anything that the latter rules out—in particular, intermingling over an interval.

We have also already seen that, unlike **HAM**, **INST** rules out intermingling on both sides of an instant, as exemplified by PREFUZZ*+POSTFUZZ*.

However, **INST** does not rule out intermingling on just one side of an instant, for example PREFUZZ*+POSTCONST, with $At(\phi, u)$. In this example, given an instant t, one of four cases holds:

- t is later than u, and hence ϕ holds at t and on any interval of the form (t, x).
- t falls within one of the intervals i_n. If n is even, ϕ holds at t and on (t, t_{n+1}); if n is odd, $-\phi$ holds at t and on (t, t_{n+1}).
- $t = t_n$ for some n. If $At(\phi, t)$ then we have either $On(\phi, (t_{n-1}, t)$ (if n is odd) or $On(\phi, (t, t_{n+1})$ (if n is even); while if $At(-\phi, t)$ then we have either $On(-\phi, (t_{n-1}, t)$ (if n is even) or $On(-\phi, (t, t_{n+1})$ (if n is odd).
- $t = u$. In this case we have $At(\phi, t)$ and $On(\phi, t, x)$ for any x.

In any case, therefore, if $At(\phi, t)$, ϕ holds on some interval limited by t, so **INST** is satisfied.

Likewise, **INST** does not rule out PRECONST+POSTFUZZ*.

6.3 TP

TP says that if ϕ is true over an interval i and false over some later interval j, then there is a *transition point* at which an interval over which ϕ is true meets an interval containing j over which ϕ is false.

6.3.1 TP is not entailed by HAM

Allen and Ferguson [2] claim that **HAM** entails **TP**. We begin by showing that this is not, in fact, correct, by constructing a counterexample in which **HAM** holds but **TP** does not.

Our counterexample is the same set-up as in the previous section, i.e., PREFUZZ* + POSTCONST. As we have seen, this satisfies **HAM**. On the other hand, **TP** does not hold, since for example, we have $On(\phi,(u,v_1))$ and $On(-\phi,i_2)$, so the antecedent of **TP** is satisfied; but there are no intervals k and l such that k meets l, $(u,v_1)\sqsubseteq l$ and $On(\phi,k)$ and $On(-\phi,l)$—given any interval k such that $On(\phi,k)$, ϕ changes truth value infinitely often between the end of k and u.

6.3.2 TP does not rule out intermingling after an instant

TP rules out the kind of intermingling in the previous example, but in fact, **TP** does not rule out all forms of intermingling at an instant. Consider, for example, the mirror image of the situation just considered, namely POSTFUZZ*+PRECONST. Suppose the antecedent of **TP** holds for some pair of intervals i,j with respect to ϕ, so we have

$$i \lhd j \wedge On(\phi,i)\wedge On(-\phi,j).$$

By POSTFUZZ*, we must have $j\sqsubseteq j_{2n-1}$ for some n, so we have

$$j_{2n}|j_{2n-1}\wedge j\sqsubseteq j_{2n-1}\wedge On(\phi,j_{2n})\wedge On(-\phi,j_{2n-1}),$$

which means that the consequent of **TP** is satisfied in this case.

Equally, suppose the antecedent of **TP** holds for i,j with respect to $-\phi$, i.e.,

$$i \lhd j \wedge On(-\phi,i)\wedge On(\phi,j).$$

In this case we know that $j\sqsubseteq j_{2n}$ for some n (we can't have $j<u$ since no such j is preceded by an i on which $-\phi$ holds). In that case we have

$$j_{2n+1}|j_{2n}\wedge j\sqsubseteq j_{2n}\wedge On(-\phi,j_{2n+1})\wedge On(\phi,j_{2n}),$$

so the consequent of **TP** is satisfied in this case as well.

6.3.3 The mirror-image of TP rules out intermingling after an instant

While **TP**, as stated, rules out the case of fuzz before an instant without fuzz after it (as exemplified by PREFUZZ*+POSTCONST), it does not rule out the case of fuzz after an instant without fuzz before it (as in POSTFUZZ*+PRECONST). This asymmetric behaviour is a natural consequence of the asymmetry in **TP** itself. To rule out POSTFUZZ*+PRECONST we should have to use the temporal mirror-image of **TP**, namely

(TPM) $$\forall i,j(i \lhd j \wedge On(\phi,i) \wedge On(-\phi,j) \rightarrow \exists k,l(k|l \wedge i \sqsubseteq k \wedge On(\phi,k) \wedge On(-\phi,l)),$$

This will rule out POSTFUZZ* + PRECONST, but of course it will not rule out PREFUZZ* + POSTCONST.

6.3.4 TP and its mirror-image together fail to rule out intermingling around an instant

The conjunction of **TP** with **TPM** will rule out both PREFUZZ*+POSTCONST and POSTFUZZ*+PRECONST; but it will not rule out intermingling at an instant altogether. To see this, consider the situation PREFUZZ*+POSTFUZZ*. This gives us intermingling in the vicinity of u, since ϕ changes truth-value at all instants t_n and v_n, and there are infinitely many such instants in any neighbourhood of u. On the other hand both **TP** and **TPM** are satisfied.

To see this for **TP**, let i and j be any intervals such that

$$i \lhd j \wedge On(\phi,i) \wedge On(-\phi,j).$$

We are looking for intervals k,l such that k meets l, j is a subinterval of l, and $On(\phi,k) \wedge On(-\phi,l)$. Since $On(-\phi,i)$, j must be contained in an interval either of the form i_{2n+1} or of the form j_{2n+1}. Let this interval be l. In the former case we can put $k = i_{2n}$; in the latter case we can put $k = j_{2n+2}$. An analogous argument goes through if ϕ and $-\phi$ are interchanged. The argument for **TPM** is just the mirror-image of the above.

An obvious idea is to try to rule out this kind of intermingling by strengthening **TP+TPM** to the following axiom:

$$\forall i,j(i \lhd j \wedge On(\phi,i) \wedge On(-\phi,j) \rightarrow \exists k,l(k|l \wedge k \sqsubseteq i \wedge j \sqsubseteq l \wedge On(\phi,k) \wedge On(-\phi,l)).$$

But obvious as the idea is, it is just as obviously not what we need, for its effect is to rule out far too much: in fact no Boolean fluent can change value more than twice in any interval if this axiom is to hold!

In fairness, it should be pointed out that **TP** was not proposed by Allen as a non-intermingling principle in itself; rather he wanted it to be a consequence of the non-intermingling principle **HAM**. Nonetheless it is instructive to examine just where **TP** stands in relation to intermingling.

6.3.5 TP does not rule out intermingling over an interval

Finally, we should consider whether **TP** rules out intermingling over an interval. In fact it does not. Consider the following set-up, using our standard t_i, u, v_j:

- for any $t < t_0$, $On(\phi,(t,t_0))$
- $Intfuzz(\phi,(t_0,u))$

– POSTFUZZ*

– for any $t > v_0$, $On(\phi, (v_0, t))$

Then in order to have

$$i \lhd j \wedge On(\phi, i) \wedge On(-\phi, j)$$

(i.e., the antecedent of **TP**) we must have $j \sqsubseteq j_{2n+1}$ for some n, and hence we have

$$j_{2n+2}|j_{2n+1} \wedge j \sqsubseteq j_{2n+1} \wedge On(\phi, j_{2n+2}) \wedge On(-\phi, j_{2n+1}),$$

so the consequent of **TP** is satisfied.

6.4 Real vs Rational Time

Up to this point we have not had to concern ourselves with the question of whether our model of the temporal order is more like the ordering of the real numbers or that of the rational numbers. When we come to consider the principles **DAV** and **FD**, however, this issue becomes crucial, and we must therefore establish some procedures for handling it first.

The key property differentiating the reals from the rationals can be expressed in the form

Every bounded set of reals has a least upper bound.

In other words, if S is a set of real numbers for which there is a real number b such that no element of S is greater than b, then there is a *least* such b:

$$\forall S \subset \mathbb{R}((\exists b \in \mathbb{R} \forall x \in S(x \leq b)) \rightarrow \exists b \in \mathbb{R} \forall b' \in \mathbb{R}((\forall x \in S(x \leq b')) \leftrightarrow b \leq b')).$$

Note that this condition also guarantees a greatest lower bound for a set which is bounded below: given such a set S, put

$$L = \{x \in \mathbb{R} \mid \forall y \in S(x < y)\}.$$

This will be non-empty so long as S is bounded below. Then L is bounded above (by any member of S), hence has a least upper bound. It is easy to see that this is also the greatest lower bound of S.

If our time-line $(T, <)$ is ordered like the reals, then, we have the (second-order) axiom

(**REALTIME**) $\quad \forall S \subset T((\exists u \forall t \in S(t \leq u)) \rightarrow \exists u \forall v((\forall t \in S(t \leq v)) \leftrightarrow u \leq v)).$

It will be useful to have an interval-based formulation of this axiom as well. The following will do (here we use *Int* to designate the set of all intervals):

(**REALTIME***) $\quad \forall Y \subset Int((\exists j \forall i \in Y(i \lhd j)) \rightarrow \exists j \forall k((\forall i \in Y(i \lhd k)) \leftrightarrow i \lhd j)).$

In $\mathbb{Q}$-time, **REALTIME** does not hold, which means that there exist instants t_r, v_s such that

$$t_1 < t_2 < t_3 < \cdots < v_3 < v_2 < v_1$$

without there being any instant u such that, for $r = 1, 2, 3, \ldots$,

$$t_r < u < v_r.$$

We shall call a configuration of this kind a *gap*, the idea being that the (irrational) instant of separation between the t_r and the v_s is 'missing' from the rational scheme of things.

An interval formulation would be that there exist intervals i_r, j_s such that

$$i_1|i_2|i_3|\cdots < \cdots|j_3|j_2|j_1$$

without there being intervals i, j such that

$$i|j \wedge \forall r(i_r \lhd j \wedge i \lhd j_r).$$

The two formulations are related by the equations $i_r = (t_r, t_{r+1}), j_r = (v_{r+1}, v_r)$.

It is worth stressing at this point that $\mathbb{Q}$-time allows some somewhat bizarre phenomena: in particular it allows a form of intermingling which differs importantly from those we have looked at up to now. For suppose we have sequences t_r, v_s as described above, and assign truth-values to ϕ according to the scheme:

$$\begin{array}{l} On(\phi, i_{2n}) \\ On(\phi, j_{2n}) \\ On(-\phi, i_{2n+1}) \\ On(-\phi, j_{2n+1}) \end{array}$$

for $n = 1, 2, 3, \ldots$. Then we have what to all intents looks like intermingling around an instant—except that there is no instant on which the intermingling centres! We shall refer to this as *intermingling around a gap*. The point about $\mathbb{R}$-time is precisely that there are no gaps in this sense.

6.5 DAV

DAV says, in effect, that every instant is the meeting point of two intervals over which ϕ has constant truth-value (the choice of the two intervals depending in general on ϕ).

6.5.1 DAV does not imply HAM

A simple counterexample suffices: let t_0 be an arbitrary instant, and assign values to ϕ by the rule

$$\forall t(At(\phi, t) \leftrightarrow t \neq t_0).$$

Then

1. For any $t < t_0$ we have $On(\phi, (t', t)) \wedge On(\phi, (t, t_0))$ where t' is any instant earlier than t;
2. For any $t > t_0$ we have $On(\phi, (t_0, t)) \wedge On(\phi, (t, t''))$ where t'' is any instant later than t;

3. For $t = t_0$ we have $On(\phi, (t', t)) \wedge On(\phi, (t, t''))$, where t', t'' are as in 1 and 2 above.

Hence **DAV** is satisfied.

On the other hand, for any $t' < t_0 < t''$, we have

$$\neg On(\phi, (t', t'')) \wedge \neg \exists j \sqsubseteq (t', t'')(On(-\phi, j)),$$

directly contradicting **HAM**.

6.5.2 DAV rules out intermingling over an interval

Let i be any interval, and let $t \sqsubseteq i$; then assuming **DAV**, there are intervals j, j' such that $sup(j) = t = inf(j')$, and $Constant(\phi, j)$ and $Constant(\phi, j')$. In particular there is an interval j such that $Lim(t, j) \wedge (On(\phi, j) \vee On(-\phi, j))$. By **Lemma1**, there is an interval $j \sqsubseteq i$ such that $On(\phi, j) \vee On(-\phi, j)$, i.e., $Constant(\phi, j)$. So **DAV** implies

$$\forall i \exists j \sqsubseteq i Constant(\phi, i).$$

This can be rewritten as

$$\neg \exists i \forall j \sqsubseteq i \neg Constant(\phi, j),$$

i.e., $\neg \exists i Intfuzz^*(\phi, i)$.

6.5.3 DAV rules out intermingling at an instant

DAVL is the formula

$$\forall t \exists u Constant(\phi, (u, t)).$$

Note that we must have $u < t$ here, since ϕ cannot have a truth-value over a non-existent interval. Now for any instant t we choose, we have

$$\begin{aligned}
\exists u < t Constant(\phi, (u, t)) &\Leftrightarrow \exists u < t(On(\phi, (u, t)) \vee On(-\phi, (u, t))) \\
&\Leftrightarrow \exists u < t(\forall t' \sqsubseteq (u, t) At(\phi, t') \vee \forall t'' \sqsubseteq (u, t) At(-\phi, t'')) \\
&\Leftrightarrow \neg \forall u < t(\exists t', t'' \sqsubseteq (u, t)(At(-\phi, t') \wedge At(\phi, t'')) \\
&\Leftrightarrow \neg \forall u < t \exists t' \sqsubseteq (u, t)(At(\phi, u) \leftrightarrow At(-\phi, t')) \\
&\Leftrightarrow \neg Prefuzz(\phi, t)
\end{aligned}$$

(The reasoning behind the penultimate step is as follows. We must have either $At(\phi, u)$ or $At(-\phi, u)$: in the former case, pick t' to be the t' of the next line, in the latter case, pick t''.) Thus **DAVL** rules out $Prefuzz(\phi, t)$ for all ϕ and t, and likewise **DAVR** rules out $Postfuzz(\phi, t)$ for all ϕ and t. So **DAV** itself rules out intermingling before, after, or around an instant.

The only form of intermingling *not* ruled out by **DAV** is the curious intermingling at a gap in $\mathbb{Q}$-time. This will be discussed further in the section on **FD**.

6.5.4 INST, TP, and TPM together imply DAV

Suppose we have **INST**, **TP**, and **TPM**, and let t be any instant, and ϕ any fluent. Since we are working in classical logic, we must have either $At(\phi, t)$ or $At(-\phi, t)$ by **NEG**.

If we have $At(\phi, t)$, then by **INST**, there is an interval i limited by t such that $On(\phi, i)$. There are two possibilities:

$$\begin{array}{ll} (1) & \exists i(t = sup(i) \wedge On(\phi, i)) \\ (2) & \exists i'(t = inf(i') \wedge On(\phi, i')) \end{array}$$

If both (1) and (2) hold, then **DAV** is immediate.

Suppose that (1) holds but not (2). Then any interval i' such that $t = inf(i')$ contains at least one instant at which $-\phi$ holds. Let u be such an instant. By **INST**, there is an interval j such that $Lim(u, j) \wedge On(-\phi, j)$.

Clearly $i \lhd j$, and we have

$$i \lhd j \wedge On(\phi, i) \wedge On(-\phi, j),$$

whence it follows by **TPM** that there are intervals k and l such that

$$k|l \wedge i \sqsubseteq k \wedge On(\phi, k) \wedge On(-\phi, l).$$

Since $At(\phi, t)$, we know that t is not in l. Hence either $t \sqsubset k$, in which case the intervals i and $(t, sup(k))$ satisfy the requirements for **DAV**, or t is the meeting point of k and l, in which case k and l will do the trick.

If we have (2) but not (1), the argument runs through similarly but with **TP** instead of **TPM**. Again, if we have $At(-\phi, t)$ instead of $At(\phi, t)$, the argument is the same but with ϕ and $-\phi$ interchanged.

In every case **DAV** is satisfied as required.

6.6 FD

FD is the principle which says that for a given fluent ϕ, any interval can be decomposed into a finite number of subintervals over each of which the truth-value of ϕ is constant.

6.6.1 FD implies DAV

This is one of the more straightforward implications we have to deal with. Suppose **FD** holds, and let $i|j$. By **FD**, we have

$$\begin{array}{rcl} i & = & i_1 \sqcup i_2 \sqcup \cdots \sqcup i_m \\ j & = & j_1 \sqcup j_2 \sqcup \cdots \sqcup j_n \end{array}$$

where for $r = 1, \ldots, m$, $Constant(\phi, i_r)$, and for $s = 1, \ldots, s_n$, $Constant(\phi, j_s)$. In particular, we have

$$i_m|j \wedge i|j_1 \wedge Constant(\phi, i_m) \wedge Constant(\phi, j_1),$$

so **DAV** is satisfied (in the form **DAVL*+DAVR***). Hence **FD** implies **DAV**; like **DAV**, therefore, it rules out intermingling over an interval and before, after, or around an instant.

6.6.2 DAV is equivalent to FD in $\mathbb{R}$-time

In view of the previous result we need only show that **DAV** implies **FD** in $\mathbb{R}$-time. Suppose then that we have both **DAV** and **REALTIME**, and let $i = (t_0, u)$ be any interval. We must show that

$$i = i_1 \sqcup i_2 \sqcup \cdots \sqcup i_n$$

where, for $r = 1, \ldots, n$, $Constant(\phi, i_r)$.

Let $T_1 = \{t \mid Constant(\phi, (t_0, t))\}$. By **DAV**, $T_1 \neq \emptyset$.

If $u \in T_1$, then we have $Constant(\phi, (t_0, u))$, so we can put $n = 1$, $i_n = i$.

If $u \notin T_1$, then T_1 is bounded above by u, so by **REALTIME** T_1 has a least upper bound, t_1.

We must have $Constant(\phi, (t_0, t_1))$, for if not, there are instants $t, t' \sqsubseteq (t_0, t_1)$ such that $At(\phi, t) \wedge At(-\phi, t')$. Now let $t'' \sqsubseteq (t_0, t_1)$, where $t < t''$ and $t' < t''$ (such an instant must exist by the density axiom T3). Then we have $\neg Constant(\phi, (t_0, t''))$, so $t'' \notin T_1$, and t_1 cannot be the least upper bound of T_1.

Repeating this construction we generate a (finite or infinite) sequence of instants

$$t_0 < t_1 < t_2 < t_3 < \cdots$$

such that for $n = 0, 1, 2, 3, \ldots$, $Constant(\phi, (t_n, t_{n+1}))$, and in addition, for $n = 0, 1, 2, 3, \ldots$, $On(\phi, t_{n+1}) \leftrightarrow On(-\phi, t_n)$.

If $t_n < u$ for all n, then $\{t_i \mid n = 0, 1, 2, 3, \ldots\}$ is bounded above by u and hence by **REALTIME** has a least upper bound, t_∞. But then we have $Prefuzz(\phi, t_\infty)$, which is ruled out by **DAV**. Hence for some n we have $t_n \geq u$. This then gives us the desired finite decomposition

$$i = (t_0, t_1) \sqcup (t_1, t_2) \sqcup \cdots \sqcup (t_{n-2}, t_{n-1}) \sqcup (t_{n-1}, u).$$

We have thus shown that in $\mathbb{R}$-time, **DAV** implies **FD**.

(This result was proved by Davis; I have not seen his proof, but it is presumably along similar lines to the above.)

6.6.3 DAV is not equivalent to FD in $\mathbb{Q}$-time

A counterexample is afforded by the 'intermingling around a gap' already described. Suppose

$$t_1 < t_2 < t_3 < \cdots < v_3 < v_2 < v_1$$

without there being any instant u such that

$$t_1 < t_2 < t_3 < \cdots < u < \cdots < v_3 < v_2 < v_1,$$

and assign truth-values according to the scheme

$$\begin{array}{l} On(\phi, (t_{2n}, t_{2n+1})) \\ On(\phi, (v_{2n+1}, v_{2n})) \\ On(-\phi, (t_{2n+1}, t_{2n+2})) \\ On(-\phi, (v_{2n+2}, v_{2n+1})) \end{array}$$

Pick an arbitrary instant t. There are four cases[5]:

- $t = t_n$ for some n. In that case, we have

$$Constant(\phi, (t_{n-1}, t)) \wedge Constant(\phi, (t, t_{n+1})).$$

- $t = v_n$ for some n. Then

$$Constant(\phi, (v_{n+1}, t)) \wedge Constant(\phi, (t, v_{n-1})).$$

- $t_n < t < t_{n+1}$ for some n. Then

$$Constant(\phi, (t_n, t)) \wedge Constant(\phi, (t, t_{n+1})).$$

- $v_{n+1} < t < v_n$ for some n. Then

$$Constant(\phi, (v_{n+1}, t)) \wedge Constant(\phi, (t, v_n)).$$

So **DAV** holds; but not **FD**, since in any interval of the form (t_i, v_j), ϕ changes value infinitely often, so no finite decomposition into subintervals of constant ϕ is possible.

6.6.4 FD rules out all forms of intermingling

Since **FD** implies **DAV**, it rules out all the forms of intermingling that **DAV** rules out, namely intermingling over an interval, and before, after or around an instant. In addition, as was shown above, it rules out intermingling around a gap. There are no other forms of intermingling to consider—one can admittedly concoct bizarre scenarios such as intermingling about each member of an infinite sequence of instants converging to a limit, but this is already ruled out by ruling out intermingling about an instant. **FD** is the *only* non-intermingling principle we have considered that rules out intermingling about a gap.

7 Concluding remarks

The results of our investigation are summarised in the following table:

	Type of intermingling				
	Over an interval	Before an instant	After an instant	Around an instant	Around a gap
INST	Impossible	Possible	Possible	Impossible	Possible
HAM	Impossible	Possible	Possible	Possible	Possible
TP	Possible	Impossible	Possible	Possible	Possible
TPM	Possible	Possible	Impossible	Possible	Possible
DAV	Impossible	Impossible	Impossible	Impossible	Possible
FD	Impossible	Impossible	Impossible	Impossible	Impossible

[5] In $\mathbb{R}$-time there would be a fifth case: t is the instant separating all the t_r from all the t_s. **DAV** would break down for this point if it existed.

We may distinguish between the 'weak' non-intermingling principles—the first four lines in the table—which only succeed in ruling out one or two forms of intermingling (but a different selection in each case), and 'strong' non-intermingling principles—the last two lines—which succeed in ruling out all or nearly all forms of intermingling.

If one seriously wants to rule out intermingling, one had better opt for one of the strong principles **DAV** and **FD**. Davis's motivation for wishing to replace **FD** by **DAV** is that **FD** 'uses a high-order expression, posed in terms of the cardinality of a set of intervals'. There is a sleight of hand here, which Davis acknowledges in a somewhat oblique fashion by his reference to non-standard models of the first-order axiomatisation of the reals. Put more directly, it would seem that to rule out intermingling *some* second-order principle is required: if you don't want to state **FD** explicitly then you have to smuggle the high-order principle in in the form of a determination to be reasoning over the reals, which require a second-order principle (e.g., the least upper bound principle) to be pinned down uniquely. But if for whatever reason you do *not* want to commit yourself to $\mathbb{R}$-time, and still want to rule out intermingling, then **DAV** is not strong enough, as we have seen: in this case, you might as well assert **FD** explicitly. The disadvantage of this is a possible loss of decidability: by Tarski's theorem, any first-order theory over the reals (which could therefore include **DAV** but not **FD**) is fully decidable.

A more exact view of the limitations of first-order theories with respect to intermingling is afforded by the Compactness Theorem for first-order logic. For suppose Σ is a first-order theory which rules out intermingling. For each positive integer n let F_n be the first-order formula

$$t_1 < t_2 < \cdots < t_{2n-1} < t_{2n} < t \wedge$$
$$At(\phi, t_1) \wedge At(-\phi, t_2) \wedge At(\phi, t_3) \wedge At(-\phi, t_4) \wedge \ldots \wedge At(\phi, t_{2n-1}) \wedge At(-\phi, t_{2n}).$$

If F_n is true for every n, we have intermingling. If there is a least upper bound for the sequence $t_1, t_2, t_3, \ldots$ then it is intermingling before that instant, otherwise it is intermingling before a gap. Since Σ rules out intermingling, we have

$$\Sigma \cup \{F_n \mid n > 0\} \models \bot.$$

By the Compactness Theorem, there must be an integer N such that

$$\Sigma \cup \{F_1, F_2, \ldots, F_N\} \models \bot.$$

But this means that in any model for Σ, the truth value of ϕ cannot change more than N times. Thus any first-order theory which successfully rules out all infinite intermingling also places a *finite* limit on the variability of each fluent[6].

Finally, a disclaimer: all the authors we have discussed, apart from Davis, have taken it for granted that we *want* to outlaw intermingling. Davis, on the other hand, is at pains to point out situations in which it might be useful to allow it—not because intermingling actually occurs in the world, but because many natural and useful idealisations of the world would be outlawed if we were not to allow it. In this paper I have confined myself to the investigation of what forms of intermingling are ruled out by each of the non-intermingling principles; I have not pursued in any depth the question under what circumstances it is desirable to rule out some or all of these forms of intermingling.

[6] This result was suggested by Ernest Davis.

Acknowledgments

I am grateful to Ernest Davis and another, anonymous, referee for suggesting a number of improvements to the originally submitted version of this paper.

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