A Qualitative Approach to Continuity

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Abstract

The standard mathematical treatment of continuity for real-valued functions presupposes a model in which arbitrarily fine distinctions can be drawn between neighbouring states. In a qualitative model these distinctions are obliterated by treating whole ranges of states as qualitatively indistinguishable. In this paper we show how topological considerations can be used, in the form of a relation called 'dominance', to handle continuity in discrete qualitative state-spaces derived from an underlying continuous space. A general theorem concerning the composition of such state-spaces is proved, and its potential usefulness illustrated by application to three different areas of spatio-temporal reasoning.

Le traitement mathématique normal de la continuité des fonctions aux valeurs réels présuppose un modèle dans lequel des distinctions arbitrairement fines peuvent être faites entre les états avoisinants. Dans un modèle qualitatif, ces distinctions sont effacées en considérant une gamme des états comme qualitativement indistinguibles.

Dans cet article, nous montrons comment utiliser les considérations topologiques, sous la forme d'une relation que nous appelons la 'dominance', afin de traiter la continuité dans les espaces des états qualitatifs, á condition que ces espaces soient dérivés d'un espace quantitatif fondamental. Nous démontrons un théorème général relatif à la composition de tels espaces, et nous éclairons son utilité potentielle par son application à trois domaines du raisonnement spatio-temporel.

1 Motivation

In this paper we attempt to develop a theory of time, change, and continuity which is consistent with the standard mathematical model that has served generations of scientists well, and which nonetheless also takes seriously the disquiet that has led many philosophers and AI researchers to seek to replace this model by something more immediately adapted to the requirements of giving a qualitative rather than quantitative description of the phenomena. This is important for any application in which physical phenomena are represented by means of general descriptions rather than in terms of numerical coordinates, as for example in AI and in natural language.

The relation between a qualitative description and the underlying continuous state-space that it is a description of can be expressed as a many-to-one mapping: each qualitative state represents a range of quantitatively distinct states. A simple example is the qualitative description of a continuous real variable as 'positive', 'negative' or 'zero'. The qualitative description 'x is positive' corresponds to an infinite set of quantitative states $\{x = r \mid r > 0\}$.

Continuity of change is usually modelled mathematically in terms of the standard condition for continuity of a *function*:

> The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$ so long as, for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every $x \in \mathbb{R}$, if $x_0 - \delta < x < x_0 + \delta$ then $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$.

But this condition can only be expressed in a model in which arbitrarily fine distinctions can be drawn between neighbouring states. In a qualitative model these distinctions are obliterated by the fact of collecting together whole ranges of states in the manner alluded to above.

None the less, continuity has important implications for the *qualitative* description of state spaces. It can constrain the kinds of change that are possible. Consider the qualitative distinction between positive, negative and zero: a variable capable of assuming any of these three descriptions may change between them. Continuity means that a change from positive to negative must involve an intervening state of zero, but this state need only hold for an instant during the transition. But if the variable changes from zero to positive and then back to zero, the positive state must hold over a period. So some qualitative states can hold for just an instant, others have to hold over periods. Given the underlying quantititative space, these facts are obvious, but the whole point about a qualitative description is that it abstracts away from the quantitative details, and vital information such as that relating to continuity is lost, unless special provision is made to retain it.

Our theory will only apply to continuous change; it is not designed to handle genuinely discontinuous change. We make certain assumptions about the enterprise of qualitative physics: the underlying world is continuous, but in keeping with the spirit of qualitative reasoning, we wish to handle propositions which describe this world without using explicit numerical parameters. In effect, the continuous and non-denumerable state-space arising from all the different possible assignments of real values to the variables is carved up into a discrete set of subspaces, within each of which all states are regarded as qualitatively the same. In doing this, important information about the qualitative character of possible changes within the state-space will be lost unless steps are taken deliberately to ensure that it is preserved. This paper presents a way of doing this.

In section 2, we outline the formal mathematical theory which is the core of this paper, culminating in the all-important composition theorem. In section 3, we discuss the application of the theory to some examples from the literature. Section 4 sketches the relationship between the discrete qualitative state spaces and the continuous spaces that underlie them. Section 5 concludes the paper.

2 Theory

In this section we outline a mathematical theory designed to handle the issues discussed in the previous section. This will form the basis for the discussion of various applications of the theory in section 3.

We begin by sketching the temporal formalism to be used in the remainder of the paper. The primitive temporal elements of our theory will be instants rather than intervals. This has been a somewhat controversial issue in AI, but we shall not attempt to address the issue directly here; the interested reader is referred to (Galton 1990, Galton 1994, Galton n.d.). Here we postulate a set $\mathcal{T}$ of temporal elements called instants, endowed with a dense and continuous strictly linear order $<$, which we may as well assume to be isomorphic to the real numbers under their usual ordering.

Intervals are defined as pairs of instants $\langle t_1, t_2 \rangle$, where $t_1 < t_2$. We denote the set of intervals $\mathcal{I}$, and will use variables i, j, k, variously decorated, to denote individual intervals. An interval is not a set of instants, though there is a natural correspondence between an interval and the set of instants which fall within it. Any interval divides the set of all instants naturally into five classes, namely the instants which precede the interval, the instant at which the interval begins, the instants within the interval, the instant at which the interval ends, and the instants which follow the interval.

If $i = \langle t_1, t_2 \rangle$ and $j = \langle t_3, t_4 \rangle$ then we write

- $Lim(t, i)$ iff either $t = t_1$ or $t = t_2$ (t **limits** i).
- $Div(t, i)$ iff $t_1 < t < t_2$ (t **divides** i).
- $Meets(i, j)$ iff $t_2 = t_3$ (i **meets** j).
- $In(j, i)$ iff $j \neq i$ and $t_1 \leq t_3 < t_4 \leq t_2$ (j is a **proper subinterval** of i).

Superimposed on our temporal theory is a theory of *states*. States are primitive elements which we notate variously S, S', S_0, etc. The primitive predicate *Holds-at* relates states to instants. The intended meaning of $Holds\text{-}at(S, t)$ is that the state S holds at the time instant t. The notion of a state holding over an interval is defined by

$$Holds(S, i) =_{\text{def}} \forall t(Div(t, i) \rightarrow Holds\text{-}at(S, t)).$$

Given two states S and S', their *State-conjunction* $S \sqcap S'$ is defined by

$$Holds\text{-}at(S \sqcap S', t) =_{\text{def}} Holds\text{-}at(S, t) \wedge Holds\text{-}at(S', t).$$

We shall say that state S' is a **perturbation** of state S if and only if at least one of the following situations can occur:

$$Holds(S, i) \wedge Lim(t, i) \wedge Holds\text{-}at(S', t)$$
$$Holds(S', i) \wedge Lim(t, i) \wedge Holds\text{-}at(S, t)$$

If only the first of the above situations can occur, then we shall say that state S' **dominates** state S, written $S' \succ S$, whereas if only the second can occur, S' is dominated by S, written $S' \prec S$. Note that, trivially, any state which can occur at all is a perturbation of itself and of any state which can hold at the same time as it; the interesting case is *proper perturbation*, where the two states are mutually incompatible. Dominance is a special case of this.

To illustrate the notions of dominance and perturbation, consider the example in which the state-space for a single real variable is divided into the three qualitative states positive, negative and zero, which we shall here abbreviate to P, N, and Z respectively. Assuming continuous variation, we have

$$Holds(Z, i) \wedge Lim(t, i) \rightarrow Holds\text{-}at(Z, t),$$

which means that neither of the situations

$$\begin{gathered} Holds(Z, i) \wedge Lim(t, i) \wedge Holds\text{-}at(P, t) \\ Holds(Z, i) \wedge Lim(t, i) \wedge Holds\text{-}at(N, t) \end{gathered}$$

can occur. On the other hand, if our variable increases monotonically over the interval $\langle -1, 1 \rangle$ so that its value is 0 at time 0, then we have

$$\begin{gathered} Holds(N, \langle -1, 0 \rangle) \wedge Lim(0, \langle -1, 0 \rangle) \wedge Holds\text{-}at(Z, 0) \\ Holds(P, \langle 0, 1 \rangle) \wedge Lim(0, \langle 0, 1 \rangle) \wedge Holds\text{-}at(Z, 0). \end{gathered}$$

It follows that Z dominates both N and P, i.e., $N \prec Z \succ P$. Equally, as we saw above, it is not possible for the value of the variable to change from N to P or vice versa without passing through Z, so N and P are not even perturbations of each other.

The motivation for the term 'dominance' is as follows. Suppose two incompatible states S and S' hold on contiguous intervals:

$$Holds(S, i) \wedge Meets(i, j) \wedge Holds(S', j).$$

Suppose also that one of S and S' holds at the instant t at which i meets j (this would necessarily be the case if S, S' were Z, P from the example above, in some order). Which of them holds? We can think of the two states as being in competition for occupying the instant t; it is the dominant state which wins.

We shall be concerned with state spaces in which all instances of proper perturbation involve dominance in one direction or other, as in the example above. The general definition is:

A **dominance space** is a pair $(\mathcal{S}, \succ)$, where

- $\mathcal{S}$ is a finite set of *states*
- $\succ$ is an irreflexive, asymmetric relation on $\mathcal{S}$, where $S_1 \succ S_2$ is read "S_1 dominates S_2",

and the following *temporal incidence rule* holds

$$\forall S, S' \in \mathcal{S}(Holds(S, i) \wedge Lim(t, i) \wedge Holds\text{-}at(S', t) \rightarrow S' \succeq S)$$

(where $S' \succeq S$ abbreviates $S' \succ S \vee S' = S$).

The key fact about dominance spaces is that a set of such spaces can be combined into a composite dominance space, as shown by the following theorem.

Theorem 1. Let $(\mathcal{S}_1, \succ_1), (\mathcal{S}_2, \succ_2), \ldots, (\mathcal{S}_n, \succ_n)$ be dominance spaces. Then

$$(\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n, \succ)$$

is also a dominance space, where $\succ$ is defined by the rule

> $(S_1, \cdots, S_n) \succ (S_1', \cdots, S_n')$ if and only if $S_i \succeq_i S_i'$ for $i = 1, \ldots n$ and $(S_1, \cdots, S_n) \neq (S_1', \cdots, S_n')$.

(Note: the ordered n-tuple $(S_1, S_2, \ldots, S_n)$ is to be understood as representing the state-conjunction $S_1 \sqcap S_2 \sqcap \cdots \sqcap S_n$.)

Proof. First, since $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ are all finite, so is $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$. Next, we must check the properties of $\succ$:

(a) $\succ$ is irreflexive. This is immediate from the condition

$$(S_1, S_2, \cdots, S_n) \neq (S_1', S_2', \cdots, S_n')$$

appearing in the definition.

(b) $\succ$ is asymmetric. Suppose that both

$$(S_1, \ldots, S_n) \succeq (S_1', \ldots, S_n')$$

and

$$(S_1', \ldots, S_n') \succeq (S_1, \ldots, S_n).$$

Then for $i = 1, 2, \ldots, n$, both $S_i \succeq_i S_i'$ and $S_i' \succeq_i S_i$, so by asymmetry of $\succ_i$, $S_i = S_i'$. It follows that $(S_1, \cdots, S_n) = (S_1', \cdots, S_n')$. Hence if we have

$$(S_1, \ldots, S_n) \succ (S_1', \ldots, S_n')$$

then we do *not* have

$$(S_1', \ldots, S_n') \succ (S_1, \ldots, S_n).$$

Finally, we must check the temporal incidence rule. Suppose that

$$Holds((S_1, S_2, \ldots, S_n), i) \wedge Lim(t, i) \wedge Holds\text{-}at((S_1', S_2', \ldots, S_n'), t).$$

Then for $k = 1, 2, \ldots, n$ we have

$$Holds(S_k, i) \wedge Lim(t, i) \wedge Holds\text{-}at(S_k', t),$$

which by the kth temporal incidence rule implies that $S_k' \succeq_k S_k$. We thus have

$$(S_1' \succeq_1 S_1) \wedge (S_2' \succeq_2 S_2) \wedge \cdots \wedge (S_n' \succeq_n S_n),$$

i.e., $(S_1', S_2', \ldots, S_n') \succeq (S_1, S_2, \ldots, S_n)$, as required. $\square$

3 Applications

We shall illustrate the theory of dominance spaces by considering three examples from the literature.

3.1 Pendula

The state of a simple pendulum can be described in terms of the anticlockwise angle θ that it makes with the vertical. A simple qualitative description might make use of just three states, namely 'left of centre' (L), 'at centre' (C), and 'right of centre' (R), corresponding to $\theta < 0$, $\theta = 0$, and $\theta > 0$ respectively (cf., (Davis 1990)). Clearly $(\{L, C, R\}, \succ)$ is a dominance space, where $C \succ L$ and $C \succ R$. In particular, it is possible to have

$$Holds(L, \langle t_1, t_2 \rangle) \wedge Holds\text{-}at(C, t_2) \wedge Holds(R, \langle t_2, t_3 \rangle),$$

showing that state C can hold instantaneously in the context of a transition from L to R; whereas it is not possible to have

$$Holds(C, \langle t_1, t_2 \rangle) \wedge Holds\text{-}at(L, t_2) \wedge Holds(C, \langle t_2, t_3 \rangle),$$

so L (and likewise R) cannot hold instantaneously, but only over an interval.

Davis (1990) considers the case of two independent pendula, A and B. Each occupies its possible states independently of the other. We thus have two independent dominance spaces $(\{L_1, C_1, R_1\}, \succ_1)$ and $(\{L_2, C_2, R_2\}, \succ_2)$, where for $i = 1, 2$ we have $C_i \succ_i L_i$ and $C_i \succ_i R_i$. Theorem 1 tells us that the combined system will also form a dominance space

$$(\{L_1L_2, L_1C_2, L_1R_2, C_1L_2, C_1C_2, C_1R_2, R_1L_2, R_1C_2, R_1R_2\}, \succ)$$

where the dominance relations are as indicated by the arrows in Figure 1. In this figure, the unbroken lines represent the 'orthogonal' transitions in which one system changes its qualitative state while the other does not: in all these cases the character of the transition is determined by the character of the transition in the system which changes. The broken lines represent the 'diagonal' transitions in which both systems undergo a change of qualitative state simultaneously. It is in determining the possible diagonal transitions that Theorem 1 really comes into its own.

We now have an immediate solution to the question considered by Davis: suppose that initially pendulum A is left of centre and pendulum B is at the centre. What are the possibilities for the succeeding state? Davis shows, using a complex set of 'Rules for Mode Transitions', that there are three possibilities: A and B both at the centre, A and B both left of centre, and A left of centre with B right of centre. These possibilities can be read off immediately from Figure 1. The problem of whether or not there can be a direct transition between two states of the complex system can be *stated*—as it was by Davis in the example just considered—without any reference to the concept of dominance. Nonetheless, it is dominance which most easily allows us to *solve* the problem.

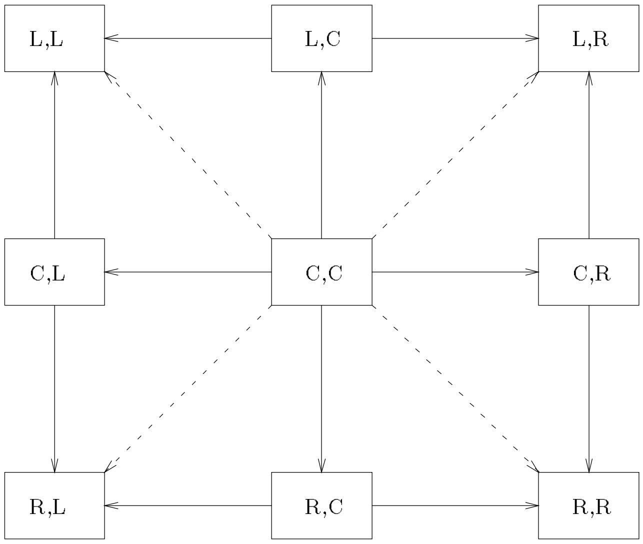

Figure 1: *Dominance relations for two independent pendula.*

3.2 Allen's interval calculus

Allen's well-known interval calculus (Allen 1984) handles temporal relationships amongst time intervals. There are 13 qualitatively different such relationships; they can be specified in terms of the relative sequence of the beginning and end points of the two intervals, as indicated in Figure 2, in which x^- and x^+ refer to the beginning and ending of the interval x, similarly with y, and the relationships presented are those that x bears to y. Entries enclosed in parentheses are superfluous in that they can be inferred from the unparenthesised entries in the same row.

It may seem strange to view the interval calculus as an example of a dynamical system: intervals cannot change *in* time, since in some sense they *are* time! However, our notion of 'perturbation' resembles Freksa's notion of *conceptual neighbourhood* (Freksa 1992), which has led to significant advances in our understanding of the Interval Calculus. In fact, the mathematical structure of the interval calculus depends entirely on the fact that intervals are regarded as bounded segments of a one-dimensional space with a favoured orientation, and has nothing to do with time as such except insofar as time can be viewed as such a space. We can treat the interval calculus as a model for many things other than temporal intervals, for example as representing *spatial* relations of objects constrained to move under only one degree of freedom.

From Figure 2, it can be seen that in all the relations apart from $<$, m, mi, and $>$ we have $x^- < y^+$ and $x^+ > y^-$, i.e., each interval begins before the other ends. The nine relations for which this holds correspond to all the possible allocations of $<$, $=$, and $>$ to the two pairs x^-, y^- and x^+, y^+. For a pair of point objects p, q, we have the simple dominance space $P = (\{p < q, p = q, p > q\}, \succ)$, where $p = q$ dominates both $p < q$ and $p > q$. In the case under consideration we have two such

	x^-, y^-	x^-, y^+	x^+, y^-	x^+, y^+
$<$	$(<)$	$(<)$	$<$	$(<)$
m	$(<)$	$(<)$	$=$	$(<)$
o	$<$	$(<)$	$>$	$<$
s	$=$	$(<)$	$(>)$	$<$
d	$>$	$(<)$	$(>)$	$<$
f	$>$	$(<)$	$(>)$	$=$
$=$	$=$	$(<)$	$(>)$	$=$
fi	$<$	$(<)$	$(>)$	$=$
di	$<$	$(<)$	$(>)$	$>$
si	$=$	$(<)$	$(>)$	$>$
oi	$>$	$<$	$(>)$	$>$
mi	$(>)$	$=$	$(>)$	$(>)$
$<$	$(>)$	$>$	$(>)$	$(>)$

Figure 2: *The 13 interval-interval relations determined by the sequence of endpoints*

spaces, namely

$$(\{x^- < y^-, x^- = y^-, x^- > y^-\}, \succ_1) \text{ and } (\{x^+ < y^+, x^+ = y^+, x^+ > y^+\}, \succ_2).$$

These spaces are independent, their Cartesian product corresponding to the nine interval relations mentioned above—assuming the conditions $x^- < y^+$ and $x^+ > y^-$ hold. Theorem 1 enables us to compute the dominance relations for the Cartesian product, and hence for nine of the interval relations, as shown in Figure 3.

$$\begin{array}{c|ccccc} & x^- < y^- & \leftarrow & x^- = y^- & \rightarrow & x^- > y^- \\ \hline x^+ < y^+ & o & \leftarrow & s & \rightarrow & d \\ \uparrow & \uparrow & \nwarrow & \uparrow & \nearrow & \uparrow \\ x^+ = y^+ & fi & \leftarrow & = & \rightarrow & f \\ \downarrow & \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ x^+ > y^+ & di & \leftarrow & si & \rightarrow & oi \end{array}$$

Figure 3: *Dominance diagram for nine of the interval-interval relations*

To complete the dominance diagram, we note that the relations $<$, m, and o only differ with respect to the pair x^+, y^-, and likewise oi, mi, and $<$ only differ with respect to the pair x^-, y^+. These two sets of three thus form dominance spaces of the form of P above, which can be attached to the nodes o and oi in Figure 3. We obtain the complete dominance space for the 13 interval-interval relations as in Figure 4. This diagram agrees as far as perturbation alone (as opposed to dominance) is concerned with Freksa's neighbourhood diagram; but unlike Freksa, we have derived it using a *general* theorem for the composition of dominance spaces.

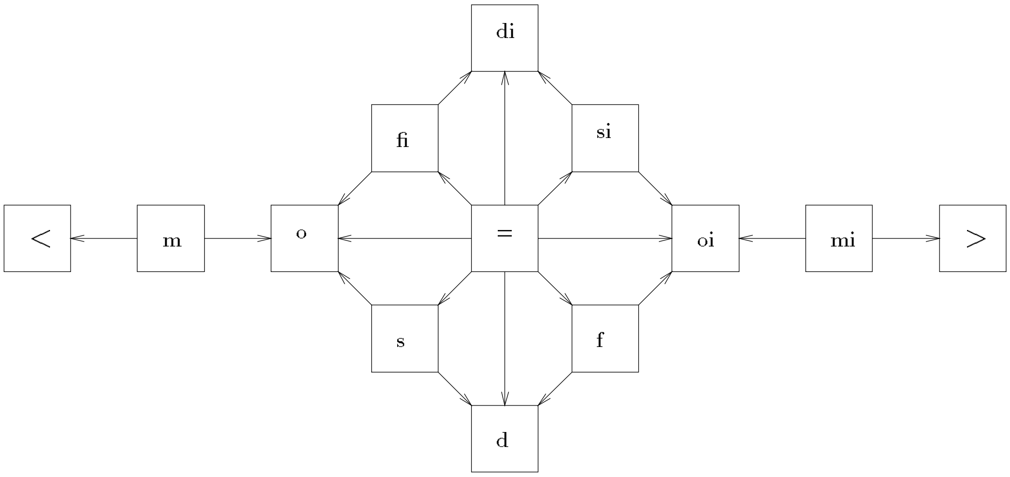

Figure 4: *The discrete space of interval-interval relations.*

3.3 Relations between spatial regions

The spatial analogue of Allen's interval calculus is the calculus of spatial relations introduced by Cohn and his associates (Randell, Cui and Cohn 1992). They consider relationships amongst spatial regions, which unlike bodies may overlap or contain each other. There are eight basic relations (hereafter called the RCC-8 set[1]), as follows:

DC	A is disconnected from B
EC	A is externally connected to B
PO	A partially overlaps B
EQ	A is equal to B
TPP	A is a tangential proper part of B
NTPP	A is a non-tangential proper part of B
TPPI	A has B as a tangential proper part
NTPPI	A has B as a non-tangential proper part

Essentially the same set of relations, derived in a different fashion, is to be found in the work of Egenhofer (Egenhofer 1991). We can construct this set of relations as a dominance space by noting that each relation is uniquely determined by knowing (a) whether all, some, or none of A is inside B, (b) whether all, some, or none of B is inside A, and (c) whether A and B share any boundary points. (Here we are using the word 'some' in its exclusive sense, i.e., some but not all). These three factors can be represented numerically as follows:

[1]The acronym RCC can be taken as standing either for the authors Randell, Cui and Cohn, or for the Regional Connection Calculus.

(a) Let

$$\alpha = \frac{\text{area of A inside B}}{\text{area of A}}.$$

Then the three states are $\alpha = 0$ (none of A is inside B), $0 < \alpha < 1$ (some of A is inside B), and $\alpha = 1$ (all of A is inside B), forming the dominance space

$$\text{none} \succ \text{some} \prec \text{all}.$$

(b) As with (a), but with A and B swapped around (so we use

$$\beta = \frac{\text{area of B inside A}}{\text{area of B}}).$$

(c) Let γ be the minimum distance between a boundary point of A and a boundary point of B. Then we have two states $\gamma = 0$ (i.e., A and B share at least one boundary point) and $\gamma > 0$ (A and B do not share any boundary point). The dominance space is

$$\gamma = 0 \succ \gamma > 0.$$

These three little dominance spaces would, if they were mutually independent, combine to give a space with $3 \times 3 \times 2 = 18$ elements. The dominance relations are, as usual, computed using Theorem 1, as shown in Figure 5.

In fact the three spaces are not independent; all but 8 of the elements are impossible (e.g., if none of A is inside B then none of B can be inside A either). The eight possible combinations correspond exactly with the spatial relations in RCC-8, as follows[2]:

	B in A	A in B	Share boundary
DC	none	none	no
EC	none	none	yes
PO	some	some	yes
TPP	some	all	yes
NTPP	some	all	no
TPPI	all	some	yes
NTPPI	all	some	no
EQ	all	all	yes

In Figure 5 the nodes corresponding to these eight possibilities are labelled with the appropriate RCC-8 designations. A clearer view of the dominance space can be obtained by deleting all the 'impossible' nodes and rearranging those that remain as in Figure 6. The diagram we have produced is exactly what we should expect on the basis of previously published work: essentially the same diagram can be found, without our dominance arrows, in (Randell et al. 1992). Once again, the novelty of our approach lies in the fact that we have derived the diagram by means of the general theorem.

The power of our approach can be further illustrated in the context of the RCC-8 calculus as follows. Suppose we are interested in the position of a movable, non-rigid

[2]We are here assuming that each region consists of a single connected component. If scattered regions are allowed, then it is possible for two regions to be PO without sharing any boundary points, e.g., if B consists of two components, one of which is TPP, and the other DC, to A.

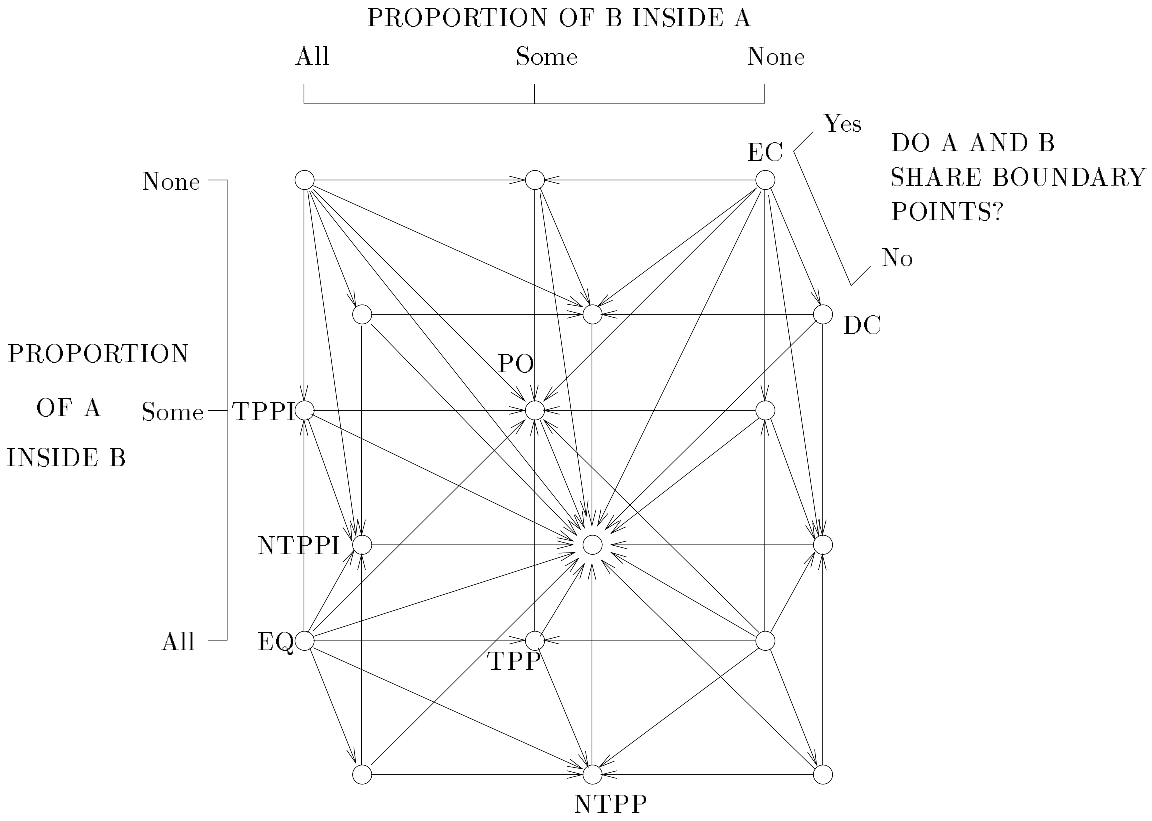

Figure 5: *Generation of the dominance space for spatial regions.*

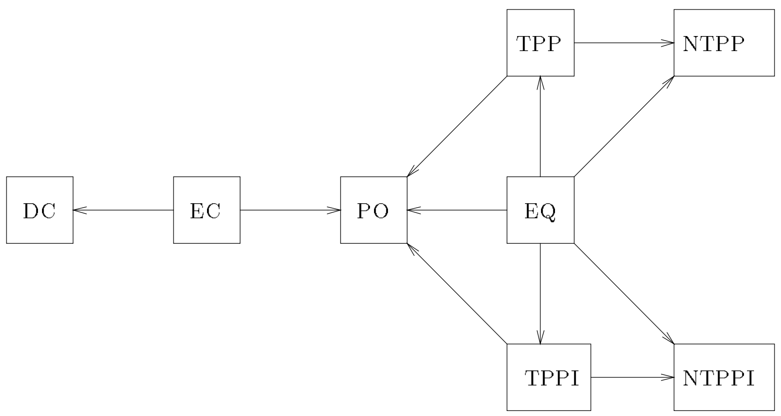

Figure 6: *The dominance space for the RCC-8 spatial relations.*

object in relation to two fixed regions R and S which are DC to each other. We are here looking at a state-space which is essentially the Cartesian product of two copies of RCC-8. As with our other examples, not every pairing represents a real possibility: for example, if the object is TPP, NTPP, or EQ to R then it must be DC to S. Once we have eliminated the impossible pairs, Theorem 1 gives us complete information about perturbation and dominance for the 31 pairs which remain. To take a specific example, consider the four composite states obtained by restricting the object to being either EC or DC to each region. Theorem 1 tells us that the dominance relations are

$$\begin{tikzcd}
(\mathrm{EC},\mathrm{EC}) \arrow[r] \arrow[d] \arrow[dr] & (\mathrm{EC},\mathrm{DC}) \arrow[d] \\
(\mathrm{DC},\mathrm{EC}) \arrow[r] & (\mathrm{DC},\mathrm{DC})
\end{tikzcd}$$

In particular, therefore, we learn that while a direct transition is possible between (EC,EC) and (DC,DC)—corresponding to the simultaneous loss of contact between the object and each of the two regions, it is not possible to pass between (EC,DC) and (DC,EC) without passing through one of the states (EC,EC) and (DC,DC). In the former case, the transition can be instantaneous (contact with R being broken at the same instant as contact is made with S), but in the latter it must take place over an interval.

4 Underlying continuous spaces

Implicit in all this work is the assumption that a dominance space is a discrete partition of what is in reality a continuum. In the examples we have looked at, the partition includes both regions and segments of their boundaries.

A planar representation of the continuous state-space for the interval calculus has been given by Ligozat and Bestougeff (1989). We give a modified form of their figure as our Figure 7. Here the position of a variable interval with respect to a fixed interval is specified in terms of two parameters: the position of its centre, and its half length. The state-space we are concerned with is the open half-plane above the horizontal axis; points on the horizontal axis itself do not correspond to intervals since the half-length parameter vanishes here. If we extend our definition of interval to include points as the limiting case of an interval having zero duration, then the horizontal axis can be seen to be partitioned into the five point-interval relations.

The dominance relations are derived from the partitioned continuum using the following rules:

- A state corresponding to an open line-segment in the continuous state-space is dominated by the state corresponding to either of its end-points.

- A state corresponding to an open region in the continuous state-space is dominated by the states corresponding to any line or point on its boundary.

These rules enable us to read off from Figure 7 the dominance-space for the interval calculus, as shown in Figure 4.

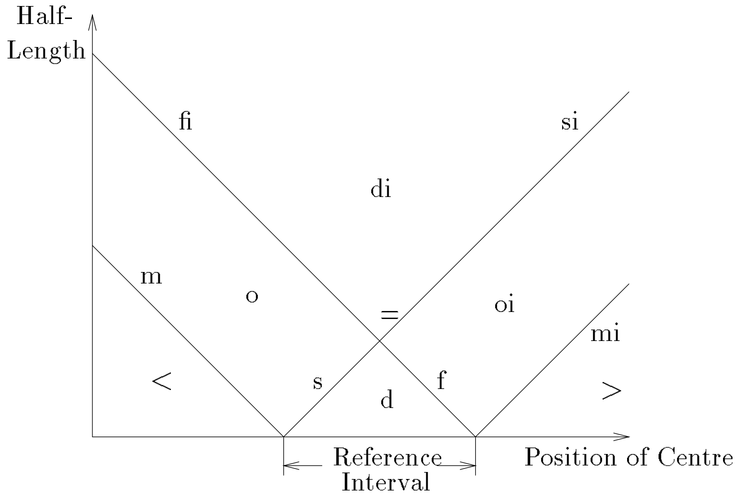

Figure 7: *The continuous space of interval-interval relations, and its partition according to Allen's interval calculus.*

For RCC-8, the underlying continuous space would be far more complicated than this if we were to insist on taking into account the exact shape and size of each region: if we did this, the space would be infinite-dimensional. If on the other hand we abstract away from exact shape and size, concentrating only on the exact values of our three variables α, β, and γ, then the state-space can be represented in three dimensions; the dominance relations can then be inferred from the topological character of those regions or boundaries within this state space which correspond to the RCC-8 relations, in much the same way as for the temporal intervals considered above.

5 Concluding remarks

In many situations, an underlying continuous system is described in terms of a discrete qualitative state-space. We have examined the structure of such state-spaces, especially features of temporal incidence for which specific provision must be made if we wish to retain them in the qualitative description. The provision we introduced, the dominance relation, concerns topological features of how the continuous state-space is partitioned into the discrete qualitative one.

Dominance is of use in relation to two kinds of problem. First, what are the possible patterns of temporal incidence in the qualitative state-space? Which states can hold at isolated instants, and in what contexts? This is what originally motivated the idea of dominance, with the observation that continuity could not be adequately modelled in a pure interval theory of time (Galton 1990). Second, how can we determine the neighbourhood relations in complex qualitative state-spaces? Theorem 1 shows how we can determine dominance, and hence neighbourhood, in spaces constructed as products of simpler spaces. Many naturally occurring spaces, e.g., the interval-interval and region-region relations, can be constructed in this way, and their neighbourhood structure straighforwardly derived.

The *results* of applying dominance in our examples are not themselves new. We advocate dominance as a general way to obtain results otherwise only to be found by *ad hoc* argumentation.

The resulting structures can be recognised as correct independently of the instant-based temporal theory used to derive them. It may therefore be possible to base the theory of dominance on a pure interval theory. It is not yet clear how to do this; but there is no reason to suppose that the theory of dominance can only be of interest to those who endorse an instant-based temporal theory.

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