# Qualitative Outline Theory

**Antony Galton** and **Richard Meathrel***
School of Engineering and Computer Science
University of Exeter
Exeter EX4 4PT, UK

## Abstract

A theory of shape is important for AI both for recognition and description of objects and for reasoning about the possible behaviours of objects. Theories of shape may be loosely classified as either volume-based or outline-based. We present a theory of the latter type, initially confined to two-dimensional outlines. We represent outlines by means of strings over an alphabet of seven qualitative curvature types, and give a regular grammar which generates the strings corresponding to possible outlines. We use subsets of the curvature-type alphabet to characterise cognitively salient subclasses of outlines, with corresponding regular subgrammars, and use decusping, smoothing, and merging operators to simplify outlines for representation at coarser granularity. We give an algorithm for deriving the curvature sequence of an outline, using only local information obtained as the outline is traversed. Finally, we indicate how more detailed (including quantitative) information can be incorporated into the theory.

## 1 Introduction

The recent upsurge of interest in qualitative spatial reasoning in artificial intelligence has given rise to prolific work in the analysis of such spatial attributes as position, orientation, and connectivity, but comparatively little seems to have been done on the equally important attribute of shape. One reason for this is surely that shape is by far the most complex, and hardest to specify, of all spatial attributes; of the aforementioned attributes, only connectivity comes anywhere near shape in the number of degrees of freedom it exhibits. The one area of artificial intelligence in which shape has been studied in some detail is Computer Vision. Here the approach tends to be rather more quantitative in nature than has been the norm in the knowledge representation community in which qualitative spatial reasoning has been developed. Outside artificial intelligence we must look to cognitive psychology for the most pertinent work on the subject of shape, some of which will be briefly discussed below.

A theory of shape is important for AI on at least two counts. First, shape plays an important part in the recognition and description of objects. This is the aspect that has been focussed on in Computer Vision. Second, since the possible behaviour of an object, particularly in interaction with other objects, is strongly constrained by its shape, reasoning about the behaviour of objects needs to be informed by a theory of shape. This reasoning may be concerned with something as simple as whether an object can fit into a container (size as well as shape is important here, of course), or whether one object can be concealed behind another, or it may be some much more complex chain of deductions concerning the behaviour of a complexly interlocking mechanism such as a clock (cf. Faltings [1990]).

Existing approaches to shape may be loosely classified as either *volume-based* or *outline-based*. A complex solid object is composed of a number of smaller solid pieces put together in a particular way, as for example a coffeepot might consist of a body in the form of a truncated cone, with a handle in the form of a half-torus and a spout in the form of a cylinder obliquely cut off at one end. This type of volume-based description is exemplified by the work of Biederman [1987] and of Marr and Nishihara [1978][1]. The alternative approach is to characterise the form of the outline of the object, for example by noting the variation in curvature across different positions. Examples of outline-based approaches to two-dimensional shape are those of Leyton [1988] and of Hoffman and Richards [1982], as well as the "multiresolution" approach of Cinque and Lombardi [1995].

In this paper we propose an outline-based approach to the classification of shape which is in certain respects more general than those just mentioned. Leyton's work, for example, assumes that the shapes being studied have smooth outlines with continuous derivatives. While any shape can be approximated by such outlines, it seems to us to be something of a limitation of

*Richard Meathrel's research is supported by EPSRC award no. 97305965.

[1] In two dimensions, "volume" becomes "area".

a shape-classification system if the only way it provides to describe a triangle, for example, is as something approximated to by a certain class of smooth curves (those having just three curvature maxima). In our system, a triangle is given an exact representation[2] which distinguishes it from any smooth curve.

## 2 Existing outline-based approaches

In this section we briefly review a number of existing outline-based approaches to shape, in order to provide a basis for comparison with the system we propose. It should be emphasised that any such scheme can be evaluated from three quite different standpoints. One of these concerns the techniques required for generating qualitative representations of shape from some antecedent "exact" representation, be it in the form of a digitised image or some continuous analogue of the shape itself. The second ("internal") point of view presupposes the existence of qualitative representations within the scheme and considers issues such as their expressive scope, manipulability, and relationships to other such representations. The third point of view is concerned with the applicability of the scheme to new or pre-existing problems. We shall not attempt a full analysis of any of the schemes from all three points of view.

### 2.1 Hoffman and Richards

Hoffman and Richards [1982] view the identification of an object as a process carried out by the visual system, in which the description ascribed to the shape of an object is used as an initial index into a stored library of shapes. They see the primary problem of shape-description as that of correctly segmenting a shape into constituent parts. The key to such segmentation is provided by the following *transversality regularity*:

> "When two arbitrarily shaped surfaces are made to interpenetrate they always [i.e., almost certainly] meet in a contour of concave discontinuity of their tangent planes."

Extrapolating from this principle, they base their object recognition system on the idea of identifying negative curvature minima on the outline of an object as points at which to segment it into parts. One problem with this, which these authors do not appear to address, is that in many cases these curvature minima alone do not determine where the segmentation lines should be drawn in the interior of the figure. Given that we know the whereabouts of the part-defining boundary points, how do we complete the segmentation?

### 2.2 Leyton

Leyton [1988] presents a theory of shape which, like that of Hoffman and Richards, accords key importance to the curvature extrema around the outline of a shape. Leyton uses curvature extrema (both maxima and minima) not to segment an object into parts, but to infer the history of processes that have acted on a shape to produce it.

The idea is that each type of curvature extremum is associated with a different kind of process in the deformational history of a shape, as follows:

| | *Maximum* | *Minimum* |
|---|---|---|
| *Positive* | Protrusion | Squashing |
| *Negative* | Internal resistance | Indentation |

With each process type is associated a *continuation rule* and a *bifurcation rule* defining the possible development of the process through time; by the table above a developmental sequence generated by these rules is translated into a corresponding sequence of outline types. By reasoning backwards one can use the current shape of an object to infer the processes which have acted on it to produce that shape.

The scope of Leyton's theory is clearly limited to the shapes of objects which do acquire their shapes through a developmental sequence of the kind it is able to describe. This primarily limits it to "natural shapes such as tumors, clouds, and embryos". It cannot apply, for example, to most human artefacts, which are typically assembled out of separate components rather than formed by a process of deformation. Another limitation is that the theory, as stated, can only apply to shapes whose outlines are smooth, continuously differentiable curves: no cusps, angles, or straight sections are allowed.

### 2.3 Cinque and Lombardi

The multiresolution approach of Cinque and Lombardi [1995] enables them to generate a sequence of strings characterising the same shape analysed at different levels of resolution, an idea that also appears in a somewhat different context in the work of Witkin [1983]. The example they give is the outline of a fish, which, at the highest resolution level, is divided into segments bearing the (cyclically-permutable) label-sequence YSCYCSCYCYCYCYCS, where 'C', 'S' and 'Y' stand for "concave", "straight", and "very convex" respectively (other possible labels being 'W' for "very concave" and 'X' for "convex"). At the lowest level of resolution shown, the string becomes XSXCXCS. The labels are obtained by simulating a heat-diffusion process on the shape, resulting in concave and convex boundary elements ending up at lower and higher "temperatures" respectively. It is essentially a quantitative technique, although the symbol strings derived from it may be regarded as qualitative in nature.

## 3 Qualitative outline theory

We offer here a system for the qualitative description of two-dimensional outlines; the system has a number of interesting and cognitively salient subsystems.

It is a formal language for shape, with interesting sublanguages. We give a regular grammar for the full language, and illustrate how grammars for the sublanguages can be derived from it.

[2] Namely, $/>/>/>$. See §3.1 for explanation.

### 3.1 Qualitative curvature types

We build our shapes from the following seven qualitative curvature types:

| | |
|---|---|
| $/$ | Straight line segment |
| $\supset$ | Convex curve segment |
| $\subset$ | Concave curve segment |
| $>$ | Outward pointing angle |
| $<$ | Inward pointing angle |
| $\succ$ | Outward pointing cusp |
| $\prec$ | Inward pointing cusp |

The shape illustrated in Figure 1 contains two straight-line segments and one of each of the other curvature types, as indicated.

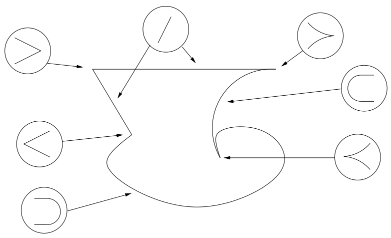

Figure 1: *The seven qualitative curvature types.*

There are a number of ways of grouping the seven types, of which perhaps the most fundamental is the separation between *linelike* elements ($/, \supset, \subset$), which contribute to the length of an outline, and *pointlike* elements ($>, <, \succ, \prec$), which do not. Another important grouping is *outward* ($\supset, >, \succ$) versus *inward* ($\subset, <, \prec$), with $/$ belonging to neither category.

A possible eighth curvature type is a *point of inflection*, where a convex curve segment meets a concave one without an angle or cusp at their meeting point. It was not felt necessary to include this since it provides no information over and above that provided by the seven types listed above.

### 3.2 Figures

A figure is defined by a cyclically permutable string of curvature-type symbols subject to the following constraints:

- The string must contain either $\supset$ or at least three convex points (to ensure boundedness).
- It must not contain two consecutive occurrences of the same curvature-type symbol.
- It must contain no two consecutive points.
- Any occurrence of either $\prec$ or $\succ$ must be adjacent (on at least one side) to an occurence of $\supset$ or $\subset$ respectively.

By "cyclically permutable" we mean that the last symbol in the string must be regarded as being followed by the first, producing a *ring* rather than a string.

We believe that these constraints are sound and complete in the sense that

1. Any "well-behaved" figure[3] has a ring description satisfying these constraints.
2. Any finite ring which satisfies the constraints defines a bounded figure (indeed a whole class of them).

The outline pictured in Figure 1 can be described, running clockwise from the bottom, by the string $\supset < / > / \succ \subset \prec$; and equally by any cyclic permutation of this string (e.g., $\succ \subset \prec \supset < / > /$) or of its reversal (e.g., $\supset \prec \subset \succ / > / <$).

### 3.3 Sublanguages

The following subsets of the set of qualitative curvature types generate important classes of outlines:

1. $\{\supset, \subset, >, <, \prec, \succ\}$ (Curvilinear outlines — i.e., no straight segments)
2. $\{\supset, \subset, /\}$ (Smooth outlines: no cusps or angles)
3. $\{\supset, /, >\}$ (Convex outlines)
4. $\{/, >, <\}$ (Polygons: no curved segments)
5. $\{\supset, /\}$ (Convex smooth outlines)
6. $\{\supset, >\}$ (Convex curvilinear outlines)
7. $\{/, >\}$ (Convex polygons)

Note that the class of convex smooth outlines is isomorphic to the class of convex curvilinear outlines: each element of the former can be converted into an element of the latter by replacing $/$ by $>$, and vice versa. Similar isomorphisms exist between other classes not listed above, e.g., "nephroids" (generated by $\{\supset, \prec\}$) and "astroids" (generated by $\{\subset, \succ\}$).

### 3.4 Qualitative outline grammars

The constraints enumerated above can be captured by means of a formal grammar which generates all and only those rings which satisfy the constraints. More exactly, for each admissible ring, the grammar will generate at least one string which represents it. The grammar, which is regular and contains 95 rules with 35 non-terminal symbols, is given in the Appendix. In this grammar, "$\Lambda$" denotes the empty string.

Formal grammars for each of the sublanguages discussed above can be derived from the full grammar by selecting an appropriate subset of the rules. For example, convex polygons, which contain only the curvature types $/$ and $>$, are generated by the rules:

$$
\begin{array}{lcl}
S & \rightarrow & /F_{/} \\
F_{/} & \rightarrow & > F_{>} \\
F_{>} & \rightarrow & /G_{/} \\
G_{/} & \rightarrow & > G_{>} \\
G_{>} & \rightarrow & /H_{/} \\
H_{/} & \rightarrow & > H_{>} \\
H_{>} & \rightarrow & /I_{/} \mid \Lambda \\
I_{/} & \rightarrow & > H_{>}
\end{array}
$$

[3] This excludes fractals and other "pathological" outlines.

which can be simplified to

$$\begin{array}{lcl} S & \rightarrow & / > / > / > H_{>} \\ H_{>} & \rightarrow & / > H_{>} \mid \Lambda \end{array}$$

Similar grammars can be constructed for the other sub-languages, with simplification possible in most cases.

The idea of a grammar for shape is not new, cf. Stiny [1980], Leyton [1988], but our grammars are not closely related to this earlier work.

### 3.5 Relations amongst the subclasses

Typically the smaller subclasses can be seen as homomorphic images of larger ones, under certain natural transformations involving a systematic loss of information. This idea is related to the phenomenon of granularity which has received considerable attention in the knowledge representation community.

From a distance, a cusp looks very much like an angle. We can therefore apply a *decusping* operation to convert every instance of $\prec$ or $\succ$ to $<$ or $>$ respectively. If we started with the full set of outlines, the resulting outlines belong to the class defined by $\{\supset, \subset, /, <, >\}$.

Other transformations are *smoothing*, by which angles are rounded off to convex curves, so that $>, <$ are replaced by $\supset, \subset$ respectively (with subsequent collapse of any resulting sequences of the form $\supset\supset$ or $\subset\subset$ to $\supset$, $\subset$ respectively), and *merging*, by which a straight section of outline is merged with an adjacent curve (thus $\supset /$ and $/ \supset$ become $\supset$, and $\subset /$ and $/ \subset$ become $\subset$, again with collapse of consecutive duplicates if necessary).

Under these transformations, an outline such as

$$\subset \succ \subset \succ / \succ$$

becomes successively transformed to $\subset > \subset > / >$, then to $\subset\supset\subset\supset / \supset$, and finally to $\subset\supset\subset\supset$, (see Figure 2).

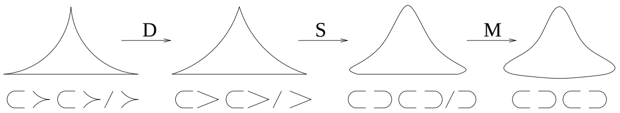


Figure 2: *Transformation of an outline by decusping, smoothing and merging.*

Figure 3 shows all the *symmetrical* subclasses, i.e., those for which each of the pairs $\{\supset, \subset\}$, $\{<, >\}$ and $\{\prec, \succ\}$ is present or absent together. The arrows indicate how the transformations of decusping, smoothing and merging operate on these subclasses. (Note that merging applied to $\{\supset, \subset, /, >, <\}$ does not give, as one might expect, $\{\supset, \subset, >, <\}$; this is because a sequence such as $> / >$ is unaffected by merging.)

### 3.6 Extracting the curvature sequence

To be useful, a representational formalism needs to be accompanied by a procedure for generating representations from the objects that they are representations of. In the case of qualitative outline theory, this means that

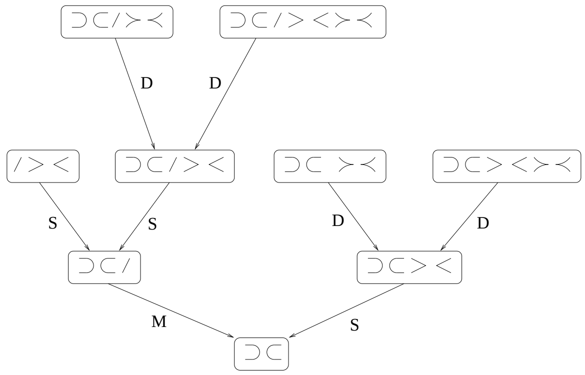


Figure 3: *Transformations on symmetrical curvature-type classes.*

we need to be able to extract the curvature sequence of an outline from the outline itself, or at any rate from a pictorial representation of it. The outline might be presented as a closed curve or colour-edge on a flat surface, or it might be the boundary of a sheet of material; we need not at this stage specify its nature in more detail. What we do assume is that we have a mechanism for tracing round the outline, keeping track of how the bearing (measured clockwise from an arbitrary fixed reference, e.g., "north") is changing. This enables us to give an algorithm for deriving the curvature sequence of an outline, using only local information obtained as the outline is traversed.

We begin anywhere we like on the outline and follow round clockwise.

- So long as the bearing varies continuously, we must have one of $/, \supset, \subset$, according as the bearing is constant, increasing, or decreasing.
- If there is a discontinuous change in the bearing, we have one of $>, <, \succ, \prec$. If the (clockwise) increase in bearing is less than 180°, we have $>$, if it is greater than 180°, we have $<$, and if it is exactly 180°, we have $\succ$ or $\prec$. To distinguish the last two cases, we have to keep track of where the section of outline just before the cusp is in relation to the section just after it. If the former is to the left as seen from the latter, we have $\prec$, if to the right, $\succ$.
- If on returning to the starting point the last curvature type is the same as the first, omit it.

The above procedure can be followed with "clockwise" and "left" swapped with "anticlockwise" and "right", and the bearings measured in the anticlockwise sense.

### 3.7 Canonical form of a sequence

In the above procedure we will obtain different results depending on where we start and whether we trace the outline clockwise or anticlockwise. But any such result can be converted to any other for the same outline by means of an appropriate cyclic permutation together with, if necessary, reversal.

To facilitate recognition and indexing of qualitatively identical outlines, we select just one of its curvature-type sequences as canonical. To do this we first establish an (essentially arbitrary—but see below) canonical ordering of the curvature types, e.g., $\supset, \subset, /, >, <, \succ, \prec$. This determines a lexicographic ordering on curvature-type sequences. For a given outline we choose that curvature-type sequence that comes earliest in this ordering. To convert an arbitrary curvature-type sequence into canonical form we must find that permutation or reversed permutation of it which is lexicographically earliest.

It should be noted that the grammar given in the appendix does not, in general, generate only the canonical curvature-type sequence for a given outline. For example, as well as the canonical string $\supset\subset\supset\subset>$, the grammar also generates the non-canonical permutations $\supset\subset>\supset\subset$, $\supset\subset\supset>\subset$, and $\supset>\subset\supset\subset$, but none of the other six possibilities. The ordering of the curvature types is not entirely arbitrary: the ordering of the three linelike elements is tailored to the way the grammar is organised, in such a way as to guarantee that the canonical sequence is always generable.

### 3.8 Quantitative considerations

It is in the nature of a qualitative representation system that one representation can correspond to many different objects. Objects which are indistinguishable within the system may nonetheless differ markedly with respect to features not accessible to the system. For a qualitative representation such as ours, this will include all the *quantitative* features whose exact expression requires the use of real-number measurements. An example of this is shown in Figure 4, where five different exemplars are given of the qualitative outline type $\supset\prec\supset\prec$. It would not, perhaps, be true to say that the differences amongst these figures are purely quantitative—but they are quantitative inasmuch as they have to do with the relative lengths and curvatures of the linelike segments. An obvious limitation of our system, as it stands, is its inability to discriminate between these different outlines.

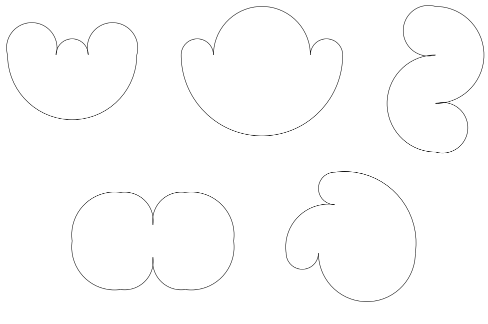

Figure 4: *Distinct exemplars of the outline type* $\supset\prec\supset\prec$.

Our qualitative representations could be extended in various ways to capture some of these distinctions. For example, each $\prec$ symbol could be annotated with indices such as $U, D, L, R$ (for "upward pointing", "downward pointing", "left pointing" and "right pointing"). Under this scheme, the first two outlines in the figure are represented as $\supset\prec_D\supset\prec_D$, the others as $\supset\prec_L\supset\prec_R$, $\supset\prec_D\supset\prec_U$, and $\supset\prec_R\supset\prec_U$. These representations are still qualitative, and allow considerable lee-way, but they certainly come closer to capturing the essential elements of the visual appearance of the outlines.

Other aspects of the outlines might be captured by annotating the linelike elements ($\supset$ in these examples) by indices denoting their relative lengths. This could be done in a "semi-qualitative" way by adopting, say, a three-point scale of "short" ($S$), "medium" ($M$) and "long" ($L$)—with an inevitable attendant arbitrariness as to the demarcations between these categories. Similarly, the angle types ($<$ and $>$) could be annotated with an indication of the size of the angle as well as the direction it is pointing in.

Clearly, our qualitative outline theory can be extended in many ways to include more precise information about the outlines it represents. It will be necessary to investigate how different such extensions interact with each other, e.g., fixing the orientations of the pointlike elements could impose constraints on the possible relative lengths of the linelike ones. The detailed working out of these constraints is likely to be problematic.

We advocate our theory as an appropriate baseline on which may be built more elaborate outline-based theories of shape to suit the particular needs of different areas of artificial intelligence and related disciplines.

## 4 Relation to other systems

Hoffman and Richards [1982] segment an outline at curvature minima, using maxima and zeros to describe the shape of each segment. This results in four basic types of segment (loosely analogous to our curvature types, but restricted to smooth curves), which they call *contour codons*[4].

Leyton [1988] classifies outlines in terms of the processes that might have given rise to them: each curvature extremum is taken as indicative of an appropriate process that has led to the formation of that extremum by acting on some initially simpler outline: "each curvature extremum implies a process whose trace is the unique symmetry axis associated with, and terminating at, that extremum". Again there is a restriction to smooth curves.

Our system does not deal with extrema except in so far as the "point-like" curvature types ($<, >, \prec, \succ$) can be so regarded; curvature extrema in smooth curves are not singled out by our system. Given that both the above systems restrict themselves to smooth continuously differentiable curves (the motivation for this being that it opens the way for a mathematical treatment using the tools of differential geometry), all the curves they consider belong to our class $\{\supset, \subset\}$, but within this class they can make finer discriminations than we can—for

[4] This approach has been extended by Rosin [1993] to include angles, cusps, and straight segments.

example all convex curves in this class are described as "$\supset$" in our system, but Leyton can recognise infinitely many varieties of this basic type according to how many protrusions and squashings they exhibit.

Cinque and Lombardi [1995] do not specify the cut-off point between "concave" and "very concave", or "convex" and "very convex", and it would seem that there is inevitably something rather arbitrary about it. Even so, however the distinctions are defined, they afford discriminations beyond the capabilities of our system.

In summary, the other systems considered here handle a restricted subset of the full range of outlines that our system handles, but on the other hand within this subset they are capable of making finer discriminations. There is a trade-off between scope and detail.

## 5 Concluding remarks

We have proposed a formal language for the qualitative representation of two-dimensional outlines. We have discussed both the scope and limitations of the language and have compared it with some other systems in the literature. We have indicated how the language may be extended, if desired, to enable finer shape-discriminations to be made. We have not addressed the problem of extending the theory to three dimensions, although this is an obvious next step to consider. This work forms part of an ongoing investigation of formalisms for the qualitative representation of shape, on which we hope to report further at a later date.

## References


[Biederman, 1987] Irving Biederman. Recognition-by-components: A theory of human image understanding. *Psychological Review*, 94(2):115–147, 1987.

[Cinque and Lombardi, 1995] L. Cinque and L. Lombardi. Shape description and recognition by a multiresolution approach. *Image and Vision Computing*, 13(8):599–607, 1995.

[Faltings, 1990] Boi Faltings. Qualitative kinematics in mechanisms. *Artificial Intelligence*, 44:89–119, 1990.

[Hoffman and Richards, 1982] D. D. Hoffman and W. A. Richards. Representing smooth plane curves for recognition. In *Proceedings of AAAI-82*, pages 5–8. American Association for Artificial Intelligence, 1982.

[Leyton, 1988] Michael Leyton. A process-grammar for shape. *Artificial Intelligence*, 34:213–247, 1988.

[Marr and Nishihara, 1978] D. Marr and H. K. Nishihara. Representation and recognition of the spatial organization of three dimensional shapes. *Proceedings of the Royal Society of London, Series B*, 200:269–294, 1978.

[Rosin, 1993] Paul L. Rosin. Multiscale representation and matching of curves using codons. *CVGIP: Graphical Models and Image Processing*, 55(4):286–310, July 1993.

[Stiny, 1980] G. Stiny. Introduction to shape and shape grammars. *Environment and Planning B: Planning and Design*, 7:343–351, 1980.

[Witkin, 1983] Andrew P. Witkin. Scale-space filtering. In *Proceedings of 8th IJCAI*, volume 2, pages 1019–1022, 1983.


## APPENDIX: A regular grammar for the full set of qualitative outlines

$$\begin{array}{lll}
S & \rightarrow & \supset \mid \supset A_{\supset} \mid \subset B_{\subset} \mid /F_{/}
\end{array}$$

$$\begin{array}{lll}
A_{\supset} & \rightarrow & /A_{/} \mid \subset A_{\subset} \mid > X \mid < X \mid \succ A_{\succ} \mid \prec X \\
A_{/} & \rightarrow & \supset A_{\supset} \mid \subset A_{\subset} \mid > X \mid < X \mid \succ A_{\succ} \mid \prec A_{\prec} \mid \Lambda \\
A_{\subset} & \rightarrow & \supset A_{\supset} \mid /A_{/} \mid > X \mid < X \mid \succ X \mid \prec A_{\prec} \mid \Lambda \\
A_{\succ} & \rightarrow & \subset A_{\subset} \\
A_{\prec} & \rightarrow & \supset A_{\supset} \\
X & \rightarrow & \supset A_{\supset} \mid /A_{/} \mid \subset A_{\subset} \mid \Lambda
\end{array}$$

$$\begin{array}{lll}
B_{\subset} & \rightarrow & /B_{/} \mid > Y \mid < B_{<} \mid \succ Y \\
B_{/} & \rightarrow & \subset B_{\subset} \mid > Y \mid < B_{<} \mid \succ B_{\succ} \\
B_{<} & \rightarrow & /B_{/} \mid \subset B_{\subset} \\
B_{\succ} & \rightarrow & \subset C_{\subset} \\
Y & \rightarrow & /C_{/} \mid \subset C_{\subset}
\end{array}$$

$$\begin{array}{lll}
C_{\subset} & \rightarrow & /C_{/} \mid > Z \mid < C_{<} \mid \succ Z \\
C_{/} & \rightarrow & \subset C_{\subset} \mid > Z \mid < C_{<} \mid \succ C_{\succ} \\
C_{<} & \rightarrow & /C_{/} \mid \subset C_{\subset} \\
C_{\succ} & \rightarrow & \subset D_{\subset} \\
Z & \rightarrow & /D_{/} \mid \subset D_{\subset}
\end{array}$$

$$\begin{array}{lll}
D_{\subset} & \rightarrow & /D_{/} \mid > W \mid < D_{<} \mid \succ W \\
D_{/} & \rightarrow & \subset D_{\subset} \mid > W \mid < D_{<} \mid \succ D_{\succ} \\
D_{<} & \rightarrow & /D_{/} \mid \subset D_{\subset} \\
D_{\succ} & \rightarrow & \subset E_{\subset} \\
W & \rightarrow & /E_{/} \mid \subset E_{\subset} \mid \Lambda
\end{array}$$

$$\begin{array}{lll}
E_{\subset} & \rightarrow & /E_{/} \mid > W \mid < W \mid \succ W \\
E_{/} & \rightarrow & \subset E_{\subset} \mid > W \mid < W \mid \succ E_{\succ} \mid \Lambda \\
E_{\succ} & \rightarrow & \subset E_{\subset}
\end{array}$$

$$\begin{array}{lll}
F_{/} & \rightarrow & > F_{>} \mid < F_{<} \\
F_{<} & \rightarrow & /F_{/} \\
F_{>} & \rightarrow & /G_{/}
\end{array}$$

$$\begin{array}{lll}
G_{/} & \rightarrow & > G_{>} \mid < G_{<} \\
G_{<} & \rightarrow & /G_{/} \\
G_{>} & \rightarrow & /H_{/}
\end{array}$$

$$\begin{array}{lll}
H_{/} & \rightarrow & > H_{>} \mid < H_{<} \\
H_{<} & \rightarrow & /H_{/} \\
H_{>} & \rightarrow & /I_{/} \mid \Lambda
\end{array}$$

$$\begin{array}{lll}
I_{/} & \rightarrow & > H_{>} \mid < H_{>}
\end{array}$$