Time and Continuity in Philosophy, Mathematics, and Artificial Intelligence

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Abstract

Both ancient and medieval philosophers were much exercised by the notion of continuity, in particular the problem of the instant of change. Aristotle held that when an object starts moving, there is neither a last instant at which it is at rest, nor a first instant when it is in motion, since being at rest and being in motion are properties which can hold of an object only over an interval, not at an instant. Modern mathematical analysis provides a different picture, based on instantaneous rates of change. The velocity of an object at an instant is the limit of its average velocity over an interval containing that instant as the length of the interval tends to zero. By the mathematical definition of continuity, when an object starts moving, there is a last moment when it is at rest, but no first moment when it is in motion.

The mathematical picture is useful in a wide range of applications, but many philosophers have found it disquieting. C. L. Hamblin suggested that for qualitative discourse we dispense with the notion of an instant altogether, and adopt a model of time in which intervals exist as entities in their own right rather than as aggregates of instants. Hamblin's approach has been duplicated in Artificial Intelligence by J. F. Allen, whose interval-based temporal framework has been widely influential. Allen's main reason for abolishing instants was that if we allow them then we have the problem of deciding whether our intervals are open or closed.

I argue that Allen was mistaken both in thinking that we can dispense with instants and in thinking that if we allow them then we have to consider open and closed intervals. A more satisfactory picture is to regard an instant as arising from the division of an interval into two parts which jointly exhaust the interval without either of them containing the instant which separates them. I present a model of time which includes both instants and intervals, without the question of closure arising, and argue that this model is the right one for the kind of qualitative simulation which has become an important desideratum of the Artificial Intelligence enterprise and which is closely related to the kinds of qualitative descriptions which Hamblin was interested in.

1 An ancient puzzle: the instant of change

The problem of the "instant of change" is an ancient one. Here, for example, is Plato in *Parmenides*:

> — But when it comes to rest after moving, and changes to being moved after resting, it is surely impossible for it to be in one time at all.
> — How so?
> — It is not possible for it to be at rest before and moved afterward, or to be moved before and at rest afterward, without changing.
> — Of course not.
> — Yes, but there is no time in which it is possible for it to be neither moved nor at rest.
> — No, there is not.
> — Furthermore, it does not change without being in the process of changing.
> — Hardly.
> — Then when does it change? Not while at rest, nor while moved, nor while in time.
> — No.

— So there is this strange thing in which it is when it is changing.

— What sort of thing?

— The instant. For the instant seems to signify something such that change proceeds from it into either state. There is no change from rest while resting, nor from motion while moving; but this instant, a strange nature, is something inserted between motion and rest, and it is in no time at all; but into it and from it what is moved changes to being at rest, and what is at rest to being moved. (Allen 1983, p.43)

When something begins to move, there is an initial period over which it is at rest, and therefore has not yet begun to move, followed by a period over which it is in motion, and therefore has already begun to move. The change itself, from rest to motion (or equally, from motion to rest), must occupy a strange durationless bit of time, an instant, that is sandwiched between the two periods. At that instant, as Aristotle argues in the following passage, the object is neither at rest nor in motion (for this quotation I use Wicksteed and Cornford's translation, which expands and clarifies Aristotle's extremely condensed argument):

> Moreover, if a stretch of time be divided by a 'now' into two periods, we have seen that this 'now' *qua* end of the first period and *qua* beginning of the second is the same. Now suppose a mobile be in motion in the whole of the first period and at rest in the whole of the second. Being in motion in the whole of the first period, it will be in motion in any of it which is naturally capable of being moved in; so that if its end were capable of being moved in, it would be moving in that. And by parity of reasoning, if the beginning of the second period were capable of being rested in, it would be at rest in the beginning of that. But the end of one and the beginning of the other are the same 'now': therefore if a 'now' were capable of being moved in and rested in, the same mobile would be at once moving in it *qua* end of one section and at rest in it *qua* beginning of the other. (*Physics*, VI, 234a35–b5)

The problem of instants was, indeed, taken up with a vengeance by Aristotle. It should be noted that ancient Greek lacked a word to denote a time instant; so Plato speaks of 'the suddenly' (*tò exaíphnēs*), while Aristotle speaks of 'the now' (*tò nūn*). Aristotle's 'now' marks the meeting place of two periods which together exhaust time and of which it is a limit or end-point:

> Part of [time] has come to be and is no longer, part is going to be and is not yet; and it is of these parts that both infinite time and any arbitrary time are composed. (*Physics*, IV, 217b33-a2)

> The now is the link of time, as we have said, for it links together the time which is past and that which is to come; and it is the limit of time, for it is the beginning of one part and the end of the other. (*Physics*, IV, 222a10)

The now is thus not *part* of time:

> The now is not a part [of time]: for the part is a measure of the whole, and a whole must consist of the parts; but time does not seem to consist of nows. (*Physics*, IV, 218a6–8)

and any two instants are separated by a period:

> Just as with points, it is surely impossible for nows to abut one another. (*Physics*, IV, 218a18–19)

Thus the instant for Aristotle has no separate existence; rather, if the temporal continuum is divided into two parts, then the instant marks the point of division. If in thought we separate the two parts then the instant can be identified with the *gap* thus produced; clearly time cannot consist only of gaps! Of course in reality the two parts are not separate, so there is no gap. Perhaps we should say that the instant represents the possibility of separating time in thought; and for each possible separation there is a distinct instant. Aristotle recognises this character of the instant as a strange 'gap which is not a gap':

> And time is made continuous by the now, and is divided at the now. (*Physics*, IV, 220a4)

Aristotle has a simple answer to the problem of whether an object is at rest or in motion at the instant when it starts moving: it is neither at rest nor in motion. This is a consequence of his more general claim that neither rest nor motion can be ascribed to an object at an instant, only over a period. Actually, Aristotle does not appear to distinguish between *moving* and *being in motion*. What he demonstrates is that an object cannot move in an instant; his *reductio* argument (at *Physics*, 234a24–32) is as follows: if an object can move in an instant then it can move faster or slower in that instant; the distance D covered in the faster motion must be greater than the distance d covered in the slower motion; so, during the faster motion it must cover the distance d during a proper part of the time it takes to cover the distance D, which we have assumed to be an instant. But an instant is indivisible, and hence has no proper parts.

Aristotle then argues that an object cannot be at rest at an instant on the grounds that it only makes sense to ascribe rest to something in a situation in which it makes sense to ascribe motion to it (*Physics*, VI, 234a33-34). Later he comes up with a separate argument:

> Moreover, we say something is at rest if it and its parts are in the same position now as previously: but in the now there is no 'previously', so there is no being at rest. (*Physics*, VI, 234b6–7)

The point here is that rest involves being in the same position from earlier to later, and hence necessarily involves reference to a period. He could have used the same argument to show that motion is not possible at an instant as well: for motion involves being in different positions earlier and later. Thus rest and motion can only be ascribed to an object over a period.

It should be said at this point that Aristotle's discussion is not as clear-cut and systematic as I have perhaps made it appear. There seems to be a considerable amount of confusion in his thought, or at any rate in the surviving texts, and commentators are not agreed on precisely what Aristotle's final view of the matter was. Nonetheless, I shall find it convenient to label the particular set of views expressed in this section as, if not Aristotle's, at any rate *Aristotelian*, to contrast them with the conflicting modern view to be discussed in the next section.

2 The view from mathematical analysis

The modern mathematical view of instants and continuity, which was worked out over the eighteenth and nineteenth centuries, differs markedly from the Aristotelian view. A good starting point is to note that we can measure durations; thus to each bounded time interval we can assign a positive number in such a way that if one interval is longer than another then the number assigned to the former is greater than the number assigned to the latter. If we choose an arbitrary instant O as origin, we can then assign to any other instant I a number as follows: if I is later than O, then assign to I the number already assigned as a measure of the duration of the interval OI; if I is earlier than O, then assign to I the number $-d$, where d measures the duration of the interval IO; and finally to O itself assign the number 0. In this way every instant gets assigned a number.

On the mathematical view, it is assumed that *there is an instant corresponding to every real number*. In partial justification of this, we note that, for example, if at instant O a point object moves at precisely one metre per second along the diagonal of a perfect square measuring 1 metre along the side, then the number assigned to the instant I at which it reaches the other end of the diagonal will be $\sqrt{2}$; if another object begins at O to move at the same speed round the circumference of a perfect circle of diameter one metre, then the instant I$'$ at which it returns to its starting point will be assigned the number π. And so on.

We thus have a one-to-one correspondence between instants and real numbers. From now on, when discussing the mathematical view, we shall simply use real numbers as names of instants: we shall talk about 'the instant $\sqrt{2}$' or 'the instant π', meaning by these expressions the instants those numbers are assigned to.

Now, what about motion and rest? For simplicity, we shall consider a rigid object which moves along a line. The line is a one-dimensional continuum, so on the mathematical view, we can represent it in the same way as we represent time, as the set of real numbers, each number representing one point on the line. The position of the object along the line at any given time can then be specified by naming the point on the line occupied by some distinguished point on the object (e.g., its rightmost point). Since at every instant the object is in a position, and at no instant is it in more than one position, the entire history of its motion along the line can be given as a function from real numbers (representing instants) to real numbers (representing positions). If the motion is continuous, then the function will be continuous too.

In most cases it seems to be a reasonable assumption that this function is not only continuous but differentiable. This means that at each instant we can assign to our object an *instantaneous velocity*, which is the limit of its average velocity over an interval containing that instant as we let the length of the interval tend to zero. (Differentiability just means that this limit exists, which is not guaranteed for an arbitrary function.) The average velocity over an interval is easily computed as the ratio of the distance covered to the duration of the interval.

We can now attack the problem of the instant of change. Suppose the object is at rest in position 0 for all $t < 0$ and in motion to the right for all $t > 0$, so it *starts* to move at time 0. Our question is: is the object at rest or in motion at time 0? To answer this, we need to define rest and motion. On the mathematical view, the definitions are as follows:

- An object is at rest if its velocity is 0.
- An object is in motion if its velocity is not 0.

We can now observe that the position of our object is given by a function f with the following characteristics:

$$\begin{array}{rcll} f(t) & = & 0 & (\text{for } t < 0) \\ f(t) & > & 0 & (\text{for } t > 0) \end{array}$$

If the motion is continuous, so f is a continuous function, then since the position is 0 at times arbitrarily close to time 0, it follows that its position must be 0 *at* time 0, i.e., $f(0) = 0$. If, further, the motion is smooth, i.e., everywhere differentiable, then it also follows that the velocity, given by the function f' (the derivative of f), obeys:

$$\begin{array}{rcll} f'(t) & = & 0 & (\text{for } t \leq 0) \\ f'(t) & > & 0 & (\text{for } t > 0) \end{array}$$

In particular, since $f'(0) = 0$, we infer that the object is at rest at the instant when it begins to move.

So far we might say that the only substantial difference between the mathematical view and the Aristotelian view lies in their differing definitions of motion and rest. Aristotle defines these qualities so that it only makes sense to attribute them to an object over an interval; the mathematical definitions allow—indeed insist—that they be attributed to an object at an instant. Now whenever, on the Aristotelian view, an object is at rest over an interval, then on the mathematical view it is at rest at every instant within that interval; and likewise with motion. But there are also instants which fall neither within a period of rest nor within a period of motion. These are the instants which separate periods of rest from periods of motion; for Aristotle it does not make sense to ascribe either motion or rest to an object then, but on the mathematical view, the object is always at rest at such instants. Thus a fundamental asymmetry between rest and motion is introduced.

The differences between the two views becomes more marked, however, when we turn from the instant of change to the further ramifications of the mathematical view. The next step is to note that an interval can be exactly characterised by the instants which fall within it, that is, the instants which divide the interval into two parts, before and after the instant. Thus the interval which begins at the instant t and ends at the instant t' can be uniquely characterised as the 'open interval' (t, t'), which is defined to be the *set* of instants $\{x \in \mathbb{R} \mid t < x < t'\}$. It is a short step from here, and a typical "mathematicians' move", to *identify* an interval with the set of instants that fall within it. This move is made so glibly that most adherents of the mathematical view do not notice that it has been made: it just seems obvious to them that an interval can be identified with a set of instants.

This now brings us into conflict with another tenet of the Aristotelian view: that an instant is not a constituent of time. If an interval is identified with a set of instants, then it comes naturally to the mathematicists (as I shall now call the adherents of the mathematical view) to think that *time consists of instants*. Actually there is a confusion here: it is not possible simultaneously to identify an interval with a set of instants and to claim that the interval is *composed* of those instants. A set is not the aggregate of its members but a mathematical abstraction to which each of its members stands in the abstract relation of membership. As to what the set *is* in itself, apart from the relationship it bears to its members, any answer will do: the matter is arbitrary and irrelevant. The mathematicist view, then, moves from the correspondence between an interval and the set of instants which divide it, to an identification of the interval with that set, and thence to an identification of the interval with the aggregate of the elements of the set.

The final step in the development of the mathematicist view is to note that in mathematics it is natural and profitable to consider not just *open* intervals of the form $\{x \in \mathbb{R} \mid t < x < t'\}$ (denoted (t, t')), but also *closed* intervals of the form $\{x \in \mathbb{R} \mid t \leq x \leq t'\}$ (denoted $[t, t']$) and *semi-open* intervals of the forms $\{x \in \mathbb{R} \mid t < x \leq t'\}$ (denoted $(t, t']$) and $\{x \in \mathbb{R} \mid t \leq x < t'\}$ (denoted $[t, t')$). Now if each open mathematical interval (t, t') corresponds to a temporal interval, then what should we say about the closed and the semi-open mathematical intervals? The mathematicists unhesitatingly assume that they too correspond to temporal intervals, so that for any given instants t and t' (where $t < t'$) we have to consider not just the interval (t, t') consisting of the instants which divide the interval, but also the three further intervals $(t, t']$, $[t, t')$, and $[t, t']$ consisting of those instants together with one or both of the instants marking the end-points of the interval.

3 Problems with the mathematicist treatment

The philosopher C. L. Hamblin discussed the instant of change problem in two papers, (Hamblin 1969) and (Hamblin 1971). He begins the former paper with the following formulation of the problem:

> At 8 a.m. I get in my car and set off for work. At 7:59 a.m., before I started it, my car was at rest; at 8:01 a.m. it is in motion. When a thing is not in motion, it is at rest, and when it is not at rest, it is in motion. But what was the state of the car *at* 8:00 a.m., as I was starting it? It would be inaccurate to say that it was in motion but it would be inaccurate, also, to say that it was at rest: it was "just starting". But, if whenever a thing is not in motion it is at rest, the state of "just starting"—or, of course, the comparable state of "just stopping"—must be either a state of motion or a state of rest. Which? (Hamblin 1969, p.410)

Hamblin does not wish to quarrel with the mathematicist account by which the instantaneous velocity at the moment of starting or stopping is precisely zero; but he contends that this is something quite different from saying that at that moment the object is at rest rather than at motion. It is the latter description, not the former, that he finds problematic:

> Our problem is concerned with the *systematic* description of physical phenomena. It arises from the clash of the continuous-variable language of Mathematical Physics with the discrete two-valued language that we would like to make work as an alternative. The mathematical description appropriate to an accelerating object does not raise problems for us; but we would like, in addition, to be able to apply the two-valued predicate 'in motion', or its negation, to any thing at any time, and this is the project that runs into difficulties. If two-valued statements are predicates of time-instants they are *essentially* discontinuous: they cannot flow smoothly from falsehood to truth, and yet our time-language presents us with a continuum on which to define them. (Hamblin 1969, p.414)

The solution that Hamblin proposes is to 'build a logic in which intervals are the primary time-elements'. They are to be 'treated as entities in their own right and not necessarily as sets of instants'.

The primitive notion of Hamblin's interval calculus is that of *abutment*. He uses the notation xAy to mean that interval x abuts interval y, that is, y follows x without a break. He can then define *precedence* using

$$x < y =_{\text{def}} xAy \vee \exists z(xAz \wedge zAy).$$

(Note that in expounding Hamblin's calculus, and quoting his formulae, I have adapted his notation to bring it into line with modern practice.) He then notes that for technical reasons it is easier to start with '$<$' as primitive and set up a definition in the reverse direction, namely:

$$xAy =_{\text{def}} x < y \wedge \neg\exists z(x < z \wedge z < y).$$

Hamblin defines other relations on intervals as follows. *Overlap* is given by

$$xOy =_{\text{def}} \neg(x < y \vee y < x);$$

and *containment* by

$$xCy =_{\text{def}} \forall z(zOx \rightarrow zOy).$$

He notes that there are altogether thirteen distinct interval–interval relations.

He then proceeds to lay down the following axioms (this is the set that appears in (Hamblin 1971), correcting a number of deficiencies in the earlier paper):

1. $\forall x \neg(x < x)$
2. $\forall w \forall x \forall y \forall z(w < y \wedge x < z \rightarrow w < z \vee x < y)$
3. $\forall x \forall y(x < y \rightarrow (xAy \vee \exists z(xAz \wedge zAy)))$
4. $\forall w \forall x \forall y \forall z(wAy \wedge wAz \wedge xAy \rightarrow xAz)$
5. $\forall w \forall x \forall y \forall z(wAx \wedge xAz \wedge wAy \wedge yAz \rightarrow x = y)$
6. $\forall x \exists y(x < y)$
7. $\forall x \exists y(y < x)$
8. $\forall x \exists y(yCx \wedge y \neq x)$
9. $\forall x \forall y(yCx \rightarrow (\phi x \rightarrow \phi y))$
10. $\forall x(\forall y(yCx \rightarrow \exists z(zCy \wedge \phi z)) \rightarrow \phi x)$

The symbol ϕ appearing in axioms 9 and 10 is meant to stand for any elementary ‘phenomenal predicate’, by which Hamblin means predicates corresponding to properties such as being at rest, or in motion, or red, and so on.

Hamblin thus provides us with a logic in which the instant of change problem does not arise, for the simple reason that *there are no instants*. It is therefore a logic that is suitable for the ‘discrete two-valued language’ of phenomenal predicates, as opposed to the continuous-variable language of physics.

Axioms 9 and 10 are of particular interest. Hamblin explains these as follows:

> Axiom 9 serves to specify of each of the elementary predicates that when it holds of an interval it holds of any subinterval: that is, “ϕa” is to be taken as asserting that ϕ is true *throughout* the interval a. To interpret axiom 10 it is convenient first to define the *negation of a predicate* in the form
>
> $$(-\phi)a =_{\text{def}} \forall x(xCa \rightarrow \neg\phi x).$$
>
> We can read “$(-\phi)a$” as “non-ϕ throughout a”, and it asserts that there is no subinterval of a throughout which ϕ holds. Now axiom 10 can be written in the alternative form
>
> $$\neg\phi a \rightarrow \exists x(xCa \wedge (-\phi)x)$$
>
> and says that whenever ϕ fails to hold throughout some interval a there is a subinterval b such that non-ϕ holds throughout it. This axiom excludes the possibility of indefinitely finely intermingled period of redness and non-redness, or of high and low pitch, or of whatever else it is that we take the elementary predicates to represent. (Hamblin 1971, pp.131–2)

Hamblin goes on to show how a logic of instants can be derived from his logic of intervals by in effect defining an instant as an equivalence class of ordered pairs of abutting intervals. That is, given xAy and zAw, we write

$$(x, y) \equiv (z, w) =_{\text{def}} xAw.$$

Then “$\equiv$” can be shown to denote an equivalence relation, and each equivalence class under this relation is identified with an instant—specifically, the equivalence class to which the pair (x, y) belongs is the instant at which x abuts y. Hamblin then asserts that the set of instants thereby defined satisfies the axioms for a dense linear order.

Hamblin comments on this construction:

> It follows that if mathematical physics needs the language of instants to deal with, say, the laws of motion it can get it eventually from an elementary logic of intervals. But we should not feel obliged, just because mathematical physics needs such a language—if in fact it does—to use it for every facet of our experience. (Hamblin 1971, p.133)

The ideas embodied in Hamblin's logic of intervals were soon taken up by a number of researchers in philosophical linguistics, for example Humberstone (1979). Unlike Hamblin, Humberstone prefers to regard the relationship between a predicate and an interval, symbolised by Hamblin in the form ϕa, as a metalinguistic relation, namely the satisfaction of a *proposition* at (or on) an interval, symbolised $\models_a \phi$.

Humberstone also remodels Hamblin's distinction between $(-\phi)a$ and $\neg\phi a$ as a distinction between two kinds of negation, *strong* negation, $\neg\phi$, and *weak* negation $\sim\phi$. In Humberstone's words:

> ... $\sim\alpha$ is true over an interval iff α fails to be true over that interval, while for $\neg\alpha$ to be true over an interval, we require not only that α fail to be true over that interval, but that α fail likewise over all of that interval's subintervals. (Humberstone 1979, p.173)

The corresponding semantic rules are:

> $\models_x \sim\alpha$ iff not $\models_x \alpha$.
> $\models_x \neg\alpha$ iff for all $y \subseteq x$, not $\models_y \alpha$.

(Here Humberstone's '$y \subseteq x$' is equivalent to Hamblin's 'yCx').

Humberstone advocates this interval-based approach as providing the right kind of semantics for applying tense logic to the kinds of propositions that we find in everyday language. Only through interval semantics, it is implied, can the tense operators of Tense Logic be brought into harmony with the tenses of natural language. This idea was not in fact new with Humberstone—it is already present, for example, in Cresswell (1977).

Tichý (1985) criticised the interval semantics movement on a number of points, of which the most telling, perhaps, is the difficulty of reconciling the notion of 'truth over an interval' with the notion of 'truth now' which alone can provide the justification for uttering a proposition. As Tichý puts it,

> One cannot say that an assertion of ... a proposition is correct just in case the couple consisting of the world which is actualized and of the interval which is current when the assertion is made is a member of the proposition [conceived as a class of world/interval pairs]. For there is no such thing as the interval current when the assertion is made. Any assertion is made in the middle of no end of current intervals. It is unclear which of them is relevant in evaluating the assertion. (Tichý 1985, p.264)

The traditional instant-based semantics avoids this difficulty precisely because *now* is an instant: there is only one 'current instant' to consider.

It is important to notice for what follows that even if the consideration of the previous paragraph decisively refutes the claims of interval semantics, Hamblin's theory is not thereby discredited. For Hamblin's intervals are part of the subject-matter referred to in his propositions, they are not part of the metalinguistic apparatus for semantically evaluating propositions which themselves contain no reference to intervals. Thus Tichý's criticism cannot touch Hamblin.

4 An artificial intelligence perspective

Every human being is in possession of a formidable knowledge and understanding of the world, and is able to reason about it in a way which, if not watertight, nonetheless passes muster in the majority of circumstances we find ourselves in. The truth of this statement only really became obvious when attempts were made to emulate this knowledge, understanding, and reasoning capacity by means of computer software: the enterprise known as Artificial Intelligence.

Humans know, for example, about space and time: we know that at any given moment a given physical body occupies a unique region of space that is disjoint from the region occupied by any other physical body with which it shares no parts. We know that to get from one region of space to another a body must traverse a continuous path between the two regions. We have some sophisticated ideas about the behaviour of physical objects in everyday situations. We know that a spherical object on an inclined plane will normally roll down the plane. We know that liquid in a container will leak out if a hole is made in the bottom of the container, and that if liquid is allowed to flow into a container without a leak then it will eventually overflow the rim of the container (Hayes 1979, Hayes 1985).

Much of our knowledge is very specifically tailored to the human situation. It goes badly wrong when we are asked to consider situations that are a little out of the way, or to consider aspects of familiar situations which go beyond what we normally need to concern ourselves with (McCloskey 1983). But such knowledge as we have is absolutely necessary for us to be able to function properly in the world we live in. One of the tasks faced by Artificial Intelligence is to codify this everyday knowledge and understanding, and to systematise the ways in which we reason about it.

Here I shall be concerned only with the part of this enterprise concerned with time and change. A persistent strand in AI research has followed what is known as the *logicist* approach. The aim is to use symbolisms derived from formal logic to represent knowledge and then to use the inferential apparatus of logic to do the reasoning. The logicist approach is particularly associated with the name of John McCarthy, who in collaboration with Patrick Hayes initiated one of the earliest attempts to handle time and change in AI, the Situation Calculus (McCarthy & Hayes 1969).

In the Situation Calculus, the complete state of the world at any instant of time is called a *situation*. If we denote the set of all possible situations by S, and the set of all time-instants by T, then the history of the world can be represented as a one-to-one function $h : T \rightarrow S$ which identifies which situations hold at which times. Broadly considered, the goal of the Situation Calculus is to define a realistic set of constraints on the possible functions h which can play this role.

In order to accomplish this, McCarthy and Hayes introduced the notion of a *fluent*, which is a function on situations (though the term 'fluent' as used by Newton refers to functions on *times*). There are *propositional fluents*, which map situations onto truth values, and *situational fluents*, which map situations onto situations. Change is introduced into the Situation Calculus by means of *actions*; we shall denote by A the set of all possible actions. The function $result : A \times S \rightarrow S$ maps an action a and a situation s onto the situation $result(a, s)$ which results from performing action a in situation s. For example, let $moving(b, s)$ be the propositional fluent which is true in just those situations s in which the body b is in motion, and let $stop(b)$ be the action by which b stops moving. Then we have

$$\forall s(moving(b, s) \rightarrow \neg moving(b, result(stop(b), s))).$$

This simply says that if something is moving and then stops, afterwards it is no longer moving.

Notice that in the Situation Calculus, fluents play a role similar to that of Hamblin's 'phenomenal predicates'; but there is a difference, in that whereas Hamblin's predicates are related directly to times (as in 'ϕa'), the fluents are related in the first instance to situations and only indirectly to times via the history function h. Thus to say that a fluent f holds true at a time t, one would have to say $f(h(t))$, which one might read as 'The situation which holds at time t has the property f'.

An obvious simplification of this, avoiding explicit mention of situations, is to say 'The property f holds at time t'. And precisely this is the style of locution favoured by the most influential successor to the Situation Calculus, namely Allen's Interval Calculus (Allen 1981, Allen 1984). Like Hamblin, Allen argues that we do not need instants; his arguments are essentially the same, though phrased somewhat differently:

> If one allows time points, one must consider whether intervals are open or closed. For example, consider the time of running a race, R, and the time following after the race, AR. Let P be the proposition representing the fact that the race is on; P is true over R, and ¬P is true over AR. We want AR and R to meet in some sense, Whether both ends of the intervals are open or closed, AR and R must either share a time point or allow time between them. Thus we have a choice between inconsistency or truth gaps, i.e., either there is a time when both P and ¬P are true, or there is a time when neither P nor ¬P is true. One solution to this problem is to stipulate by convention that intervals are open at the lower end and closed at the upper end, but then every interval has only a single endpoint. The artificiality of this solution reinforces the argument against allowing points. (Allen 1984, p.128)

Note that this argument assumes that 'allowing time points' means adopting the mathematicist view of an interval as a set of points, although all Allen says in support of this assumption is that 'a good analogy ... is that times correspond to intervals on the real line'.

Apart from the substantial differences in notation, a great deal of Allen's interval calculus could have been lifted bodily from Hamblin's logic of intervals—though as a matter of fact it appears to have been an independent development. Like Hamblin (and in contrast to McCarthy and Hayes), Allen opts for intervals as the basic temporal entity, deriving instants as equivalence classes of intervals in a way which, though different from Hamblin's, is ultimately equivalent to it (Allen & Hayes 1985). Like Hamblin, Allen takes abutment as the primitive interval–interval relation, written $meets(i, j)$, and he notes that there are thirteen such relations altogether, which can all be derived from $meets$. Moreover, the axioms given by Allen lead to a system entirely equivalent to that of Hamblin (Galton 1990).

To say that property p holds on interval t, Allen writes '$holds(p, t)$', and this corresponds to Hamblin's notation 'ϕa'. Amongst his axioms Allen includes H2:

$$\forall x(holds(p, x) \leftrightarrow \forall y(in(y, x) \rightarrow \exists z(in(z, y) \wedge holds(p, z)))).$$

Translated into Hamblin's notation this is

$$\forall x(\phi x \leftrightarrow \forall y(yCx \rightarrow \exists z(zCy \wedge \phi z))).$$

It is easy to see that this implies Hamblin's axiom 10, and only a little more complicated to show that it implies his axiom 9.

Like Hamblin, Allen has no problem about the instant of change. An object can be at rest over interval i and in motion over interval j, where i meets j, and the question whether it is at rest or in motion at the instant where i meets j simply does not arise: there is, says Allen, no need to consider the instant at all. And if one nonetheless insists that one *must* be able to consider the instant, because of the equivalence class construction, Allen could quite well reply that, still, there is no need to consider which property, motion or rest, holds then.

It should be noted that Allen's notation has the advantage over Hamblin's that it can be readily extended to allow one to talk about *events* as well as properties or states. To say that an event e occurs over the interval t, Allen writes $occurs(e, t)$. I mention this here because it is important for a logic of temporal discourse to handle events properly, although for our present purposes we shall have no further need to refer to this.

5 Problems with the artificial intelligence view

None of the authors considered above seems to have noticed that banishing instants creates new problems. The trouble is that by ruling out instants we thereby rule out continuous change: for a continuously varying quantity does not maintain its value over any interval, however short, so the time at which it has any particular value has to be an instant. To Hamblin's argument:

> The red book on my desk could turn green for half a second or half a century but it could not turn green temporarily and durationlessly at the stroke of twelve, remaining red at all times earlier and later. (Hamblin 1969, p.414)

my response is

> [Hamblin] does not seem to notice ... that the book could instantaneously be a *particular shade* of green, passing through it on the way from blue to yellow, say. The reason why it cannot be instantaneously green, *tout court*, is that green consists of an extended continuum of different particular shades. In the same way, a moving point may be located instantaneously at a point in space, but it could not be located instantaneously within a region. (Galton 1984, p.19)

Developing this point further, in a critique of Allen's Interval Calculus, I argued in (Galton 1990) essentially as follows. Suppose a body B moves uniformly from position P to position R via postion Q over time interval I. Then there must be intervals I_1 and I_2 such that

1. B moves from P to Q over interval I_1
2. B moves from Q to R over interval I_2
3. I_1 abuts I_2
4. I_1 is an initial segment of I
5. I_2 is a final segment of I

All of these statements are, I take it, uncontroversial on anybody's theory, and in particular can readily be expressed in the Interval Calculus. In addition, intuition tells us that B is at Q at the instant where I_1 meets I_2; but Allen cannot say this! In fact, (1)–(5) above imply that B is not at Q over any subinterval of I; in Allen's notation:

$$\forall j(In(j,i) \rightarrow \neg Holds(at(b,q),j)).$$

But by Allen's definition of negation, this implies

$$Holds(not(at(b,q)),i).$$

In other words, throughout I, B is not at Q; and this conflicts with the requirements of intuition.

The problem with both Allen's and Hamblin's theories is that they are so set up that for a property to hold at all it has to hold over an interval. For an object to be in a position, it has to be in that position over an interval. But if it is in one position over an interval, then it is at rest in that position then. So according to Hamblin and Allen, an object can only be in a position by being at rest there. But this has the consequence that continuous motion is impossible: instead, all motion must consist of jumps by which the moving object passes from one position to another without traversing the intervening space. Here again we touch on an ancient subject; for a possible formulation of Zeno's arrow paradox is surely something like this:

Premisses
Motion, if it occurs at all, must be continuous.
Whenever an object occupies a position then it is at rest in that position.
Conclusion
Motion is impossible.

It might be said that, at least as regards a criticism of Hamblin, these examples are unfair. Hamblin was, after all, explicit about the distinction between continuous real-valued variation and discrete two-valued variation. Hamblin's logic of intervals was motivated by the fact that the mathematical model that is appropriate to the former leads to problems when used as a basis for the latter. This objection will not hold water, however, since for every real-valued function there is a set of two-valued predicates expressing that the function takes on this or that particular value. Corresponding to the real variable H, for example, which gives the height of a ball above the ground at any given time, there is a set

$$\{H = h \mid h \in \mathbb{R}\}$$

of propositional variables each of which is at any given time either true or false. Any one of these is on all fours with Hamblin's 'phenomenal predicates' except with respect to the crucial issue of whether or not it can be true instantaneously.

It becomes necessary, therefore, to classify two-valued predicates according to whether or not they possess this particular property. In (Galton 1990) I made a first shot at this. I defined a *state of position* to be a phenomenal predicate which *can* hold instantaneously, and indicated that for such a predicate it was reasonable to insist that if it holds over some interval then it must be regarded as holding at the instants which limit that interval. A *state of motion*, by contrast, can only be regarded as holding at an instant by virtue of the fact that it holds over some interval containing that instant. Thus, given any continuously-varying quantity q, the statement that q has a particular value q_0 will express a state of position, whereas the statements that q is changing, and that it differs from q_0, express states of motion.

I went on to develop a logic of states of position and states of motion, very much in the spirit of Allen's Interval Calculus though with a temporal basis that included instants as well as intervals. This temporal basis was not of the mathematicist kind, though, since intervals were not sets of instants; rather intervals were characterised much as in Allen's theory, with instants defined as the limits of intervals. I further showed that Allen's (and therefore Hamblin's) system corresponded exactly to that fragment of my system which concerned states of motion: all Allen's properties, and all Hamblin's phenomenal predicates were, in my terminology, states of motion. And as regards our body B moving from P to R via Q, my system gives the desired result that B is at Q at the instant at which I_1 abuts I_2, but at no other time in the interval I.

The system of states of position and states of motion solves the instant of change problem as follows. It is a consequence of the definitions that the negation of a state of motion is a state of position, and *vice versa*—from which, incidentally, it follows that Allen's $not(P)$ is not the true negation of P, since both behave like states of motion. Hence whenever we have state S holding over an interval I_1 and then its negation holding over I_2, where I_1 abuts I_2, either S is a state of position and its negation is a state of motion, in which case S holds at the instant of change; or S is a state of motion and its negation is a state of position, in which case S does not hold at the instant of change.

Now, as regards the specific case of motion and rest, clearly motion is a state of motion; if we wish to regard rest as precisely the negation of motion, then it must be a state of position. This leads, in agreement with the mathematicist model, to the conclusion that an object is at rest at the instant that it starts or stops moving. As we have seen, Aristotle's view of rest was different: for Aristotle, if it makes sense at all to say that an object is at rest at an instant, this can only be by virtue of its being at rest over a period containing that instant. Characterised thus, rest is actually a state of motion! (Remember that 'state of motion' is a term of art: there is no real paradox here.) For Aristotle, it is not possible for an object to be at rest for an isolated instant: and indeed, in the cases of a pendulum at the end of its swing and a vertically thrown ball at the topmost point of its trajectory—cases where we would naturally want to say the body was instantaneously at rest—Aristotle is forced to say that there is a short delay between the object's arriving at its extremal point and its departure therefrom:

> For if body E moves up to point D and then turns back and moves down again, the point D at which it turns serves as both end-point and beginning, the one point functioning as two. Therefore it must stop, for it has not simultaneously come to be at D and ceased to be at D, for that would mean that it simultaneously was and was not there at the same instant. (*Physics*, VIII, 262a)

6 A new model

More recently I have discovered a problem with the simple two-fold classification into states of position and states of motion. The following example will illustrate what I mean. Let A, B, and C be three billiard balls on a table-top, and consider the *qualitative* description of their relative positions. All we are interested in is which balls, if any, are in contact with each other. Since each pair can be in contact or separated independently of either of the other two pairs, it follows that there are $2^3 = 8$ distinct configurations. Writing XY to indicate contact between balls X and Y, and $\Box$ to indicate that no ball is in contact with any other, the eight configurations are

1. $\Box$
2. AB
3. AC
4. BC
5. $AB + AC$
6. $AB + BC$
7. $AC + BC$
8. $AB + AC + BC$

It is obvious that $\Box$ is a state of motion, since if all three balls are separated they must remain separated over an interval—this is because in order to get from $\Box$ to one of the other configurations at least one pair of balls must move into contact, and this will take time, during which $\Box$ continues to hold. Similarly,

$AB + AC + BC$ must be a state of position: for it can hold instantaneously, and it must hold at the limits of any interval over which it holds.

Now consider one of the other states, say AB. Is this a state of motion or a state of position? It seems to have some of the properties of both. For on the one hand it can hold instantaneously: suppose in state $\Box$, A and B come together and immediately move apart; then state AB holds just for an instant. On the other hand, its *negation* can hold instantaneously: for suppose in state AB, C comes instantaneously into contact with B and then moves away again; then $AB + BC$ (and therefore the negation of AB) holds just for an instant. So it looks as though AB behaves like a state of position in relation to $\Box$ but like a state of motion in relation to $AB + BC$.

It has therefore become necessary to replace the absolute notion of states of motion *vs* states of position by a *relative* notion which I call 'dominance'. Given two incompatible states S and S', we say that S *dominates* S' if it is possible to have S holding at an instant which limits an interval over which S' holds, but not *vice versa*. Then in particular, if S and S' hold over contiguous intervals, it is the dominant S which holds at the instant where the two intervals meet.

In our three-balls example, AB dominates $\Box$ and is in turn dominated by $AB + AC$ and $AB + BC$. It neither dominates nor is dominated by BC: any time at which BC holds must be separated from a time at which AB holds by a time at which some other configuration holds. We say that BC is not a *perturbation* of AB. For the full set of perturbation and dominance relations for this example, see (Galton 1993), where the whole theory is presented in some detail.

What then does our final theory look like?

We shall follow Hamblin in taking intervals as primitive and constructing instants as equivalence classes of abutting pairs of intervals. Following Hamblin, we shall use lower-case Roman letters to denote intervals, and lower-case Greek letters to denote instants. In addition, if i abuts j we shall write $\delta(i, j)$ to denote the instant at which they meet, i.e., the equivalence class containing (i, j).

I have no quarrel with Hamblin's axioms 1–8; as Hamblin indicated, these suffice to endow the set of instants with the properties of a dense linear ordering. This does not quite give us the mathematicist picture of instants as possessing the ordering properties of the set of real numbers. There must, to be sure, be infinitely many instants, but there is no reason why this should not be a countable infinity, with the ordering properties of the rational numbers. To get indenumerably many instants we need indenumerably many intervals; one way of ensuring this would be to have a second-order axiom to the effect that for any set I of intervals there is an interval s such that an arbitrary interval overlaps s if and only if it overlaps at least one member or I (so s is the "join" of all the members of I):

$$\forall I \exists j \forall k (kOj \leftrightarrow \exists i \in I(kOi)),$$

where I ranges over arbitrary sets of intervals—this axiom fails for open intervals on the rational line, for example, since e.g., the union of the set $\{(p, q) \mid p < q, q^2 < 2\}$ of such intervals is not itself an interval of this type.

Of particular interest are the relations between instants and intervals:

- *For every interval there are instants marking its beginning and end-points.* This follows from axioms 3, 6, and 7. Thus 3 and 6 together imply that for any interval i there is an interval j such that jAi. The instant $\delta(j, i)$ marks the beginning of i. Likewise 3 and 7 imply that for any i there is a k such that iAk; and the instant $\delta(i, k)$ is the endpoint of i. We shall denote these endpoints $\delta^-(i)$ and $\delta^+(i)$ respectively.

- *For any pair of distinct instants there is an interval of which they are the endpoints.* Let κ, λ be distinct instants, and let $(k_1, k_2) \in \kappa$ and $(l_1, l_2) \in \lambda$. By axiom 2, since $k_1 < k_2$ and $l_1 < l_2$, either $k_1 < l_2$ or $l_1 < k_2$. Without loss of generality we may suppose the former. On the other hand, it is not the case that $k_1 A l_2$ (else $\kappa = \lambda$), hence by axiom 3 there is an interval j such that $k_1 A j$ and $j A l_2$. But then we have $\kappa = \delta^-(j)$ and $\lambda = \delta^+(j)$, so the given instants are the two endpoints of j.

We introduce a predicate *Div* to express the idea of an instant's falling within, or dividing, an interval. Specifically:

$$Div(\iota, i) =_{\text{def}} \delta^-(i) < \iota \wedge \iota < \delta^+(i).$$

The mathematicist view would then be to equate i with $\{\iota \mid \mathit{Div}(\iota, i)\}$, a move which in itself is harmless, but which all too readily leads on to the idea that one can add the endpoints of the interval to obtain an interval which is somehow different, and then the to my mind untenable notion that the duration of an interval is somehow built up out of the aggregation of durationless components. We shall thus eschew the initial identification of an interval with the set of instants which divide it.

In addition we introduce predicates to express the idea of an instant's bounding, or limiting, an interval:

$$\begin{aligned} \mathit{Lim}^-(\iota, i) &=_{\text{def}} \quad \iota = \delta^-(i) \\ \mathit{Lim}^+(\iota, i) &=_{\text{def}} \quad \iota = \delta^+(i) \\ \mathit{Lim}(\iota, i) &=_{\text{def}} \quad \mathit{Lim}^-(\iota, i) \vee \mathit{Lim}^+(\iota, i) \end{aligned}$$

We must now consider Hamblin's phenomenal predicates—Allen's properties. I shall refer to these as *states*. In order to allow continuous change, we must allow some of these to hold at isolated instants; *a fortiori*, therefore, it must make sense to speak of a state holding at an instant. In fact, we shall allow this to make sense for *any* state, not just those which can hold for *isolated* instants. If a state can hold both at an instant and over an interval, then Hamblin's form of representation using predicates becomes a little awkward, since any such predicate would have to be able to take both instants and intervals as arguments. Allen's $Holds$ points the way: we distinguish two forms of holding: holding at an instant, and holding on (or over) an interval. This gives us the notations *Holds-at* and *Holds-on* introduced in (Galton 1990). Thus given a state s, an interval i, and an instant ι, we can form atomic propositions $\mathit{Holds\text{-}on}(s, i)$ and $\mathit{Holds\text{-}at}(s, \iota)$.

We shall allow that any state may be said to hold at an instant if it holds on an interval divided by that instant:

$$\forall \xi (\exists x (\mathit{Div}(\xi, x) \wedge \mathit{Holds\text{-}on}(s, x)) \rightarrow \mathit{Holds\text{-}at}(s, \xi));$$

and likewise that it holds on an interval if it holds at every instant which divides that interval:

$$\forall x (\forall \xi (\mathit{Div}(\xi, x) \rightarrow \mathit{Holds\text{-}at}(s, \xi)) \rightarrow \mathit{Holds\text{-}on}(s, x)).$$

Note that the first of these formulae entails the converse of the second, so that together they give

$$\forall x (\mathit{Holds\text{-}on}(s, x) \leftrightarrow \forall \xi (\mathit{Div}(\xi, x) \rightarrow \mathit{Holds\text{-}at}(s, \xi))).$$

However, we must be careful not to insist on the converse of the first formula, for that would make it impossible for s to hold at an isolated instant.

Instead we must specify in more detail what different kinds of state there are. Suppose we have states s and s' for which the following is possible:

$$\mathit{Lim}(\iota, i) \wedge \mathit{Holds\text{-}at}(s, \iota) \wedge \mathit{Holds\text{-}on}(s', i).$$

Then we shall say that s is a *bounding state* for s'. For example, a body's having velocity zero is a bounding state for its having positive velocity, but the latter is not a bounding state for the former. Two bodies' being in contact is a bounding state for their being separated, but not vice versa. If s and s' are compatible, so that they can hold simultaneously, then each is trivially a bounding state for the other.

If either s is a bounding state for s' or s' is a bounding state for s, we say that s and s' are *perturbations* of each other. If s is a bounding state for s' but s' is not a bounding state for s then we say that *s dominates s'*.

It follows from these definitions that 'being a bounding state of' is a reflexive relation, that 'being a perturbation of' is reflexive and symmetric, and that dominance is irreflexive and asymmetric. None of the three relations is transitive.

The negation $-s$ of a state s is defined to be the state such that

$$\forall \iota (\mathit{Holds\text{-}at}(-s, \iota) \leftrightarrow \neg \mathit{Holds\text{-}at}(s, \iota)).$$

A *state of position* is a state which dominates its negation. Suppose s is such a state, and suppose we have intervals i and j such that

$$iAj \wedge \mathit{Holds\text{-}on}(s, i) \wedge \mathit{Holds\text{-}on}(-s, j).$$

Either *Holds-at*$(s, \delta(i, j))$ or *Holds-at*$(-s, \delta(i, j))$ must be true; which? In the former case, since *Lim*$(\delta(i, j), j)$, s would be a bounding state for $-s$, whereas in the latter, since *Lim*$(\delta(i, j), i)$, $-s$ would be a bounding state for s. Only the former case is compatible with s being a state of position. Hence we have, for a state of position s,

$$\forall x \forall y(xAy \wedge \textit{Holds-on}(s, x) \wedge \textit{Holds-on}(-s, y) \rightarrow \textit{Holds-at}(s, \delta(x, y))),$$

which solves the instant of change problem for the case where a state of position changes into its negation.

For change in the opposite direction, it is equally straightfoward:

$$\forall x \forall y(xAy \wedge \textit{Holds-on}(-s, x) \wedge \textit{Holds-on}(s, y) \rightarrow \textit{Holds-at}(s, \delta(x, y))),$$

For a state s which is neither a state of position nor the negation of a state of position, the assertion that

$$iAj \wedge \textit{Holds-on}(s, i) \wedge \textit{Holds-on}(-s, j)$$

does not provide enough information to determine whether or not *Holds-at*$(s, \delta(i, j))$. If we define the *disjunction* of states by

$$\textit{Holds-at}(s \sqcup s', \iota) =_{\text{def}} \textit{Holds-at}(s, \iota) \vee \textit{Holds-at}(s', \iota),$$

then in general the negation of a state s can be expressed as a disjunction of three states

$$-s = d^- \sqcup d_0 \sqcup d^+,$$

where s dominates d^-, is dominated by d^+ and is not even a perturbation of d_0, and the three disjuncts are pairwise incompatible. In order to determine whether *Holds-at*$(s, \delta(i, j))$ in the situation described at the beginning of this paragraph, we need to know which of d^-, d_0 and d^+ holds over j or some initial segment thereof. If d^- holds over j, then s must hold at $\delta(i, j)$; if d^+ holds over j then d^+ must hold at $\delta(i, j)$ as well. What if d_0 holds over j? In that case neither d_0 nor s can hold at $\delta(i, j)$, since neither is a bounding state for the other. What holds at $\delta(i, j)$ must be a bounding state for both s and d_0. Of the four mutually exclusive and exhaustive set of states s, d^-, d_0, and d^+, only d^+ can have this property, though it is not guaranteed to. Only if it does is the scenario envisaged possible.

For example, suppose in the three balls scenario that AB holds over i and $-AB$ holds over j. In this case we have

$$\begin{aligned}
d^- &= \square \\
d_0 &= AC \sqcup BC \sqcup (AC + BC) \\
d^+ &= (AB + BC) \sqcup (AB + AC) \sqcup (AB + AC + BC)
\end{aligned}$$

Assume that d_0 holds over an initial segment of j. Then one of AC, BC, or $AC + BC$ must hold over an initial segment of j. For each of these states we must find a bounding state which is also a bounding state for AB. The states required are $AB + AC$, $AB + AC + BC$ and $AB + AC + BC$ respectively. Note that, as predicted, these belong to d^+.

We thus have a mechanism for solving the instant of change problem even in complex cases, so long as we understand the dominance relations amongst the various elements of the state-space.

7 Conclusions

We have considered the problems that arise when we try to give qualitative descriptions of continuous change. The problem of the instant of change is of great antiquity, and we outlined a number of different approaches to it. In particular, we contrasted the 'mathematicist' view, which models time by identifying each instant with a real number and each interval with a set of instants, with the approaches of Allen and Hamblin, which take intervals as the primary entities, deriving instants only secondarily from them.

We saw that the interval-based approaches, as presented by Allen and Hamblin, do not allow for continuous change. This is because they have nothing to say about when and whether a state can be said to be hold at an instant. We therefore advocate a system in which states can hold at instants as well as intervals. This forces one to distingish different kinds of states according to the circumstances, if any, in which they can hold at isolated instants or at the limits of intervals within which they do not hold. The way to do this is by considering a state as a member of a state-space over which special relations of perturbation and dominance are defined. Once this is done, it becomes possible to determine the possible patterns of temporal incidence for all of the states in the space.

The kind of state-space referred to in the previous paragraph can be regarded as a qualitative projection or homomorphic image of the continuous, quantitative space that exists in reality. Our enterprise is thus closely connected to the branch of AI known as 'qualitative physics' in which, to quote Forbus (1988), the key idea is 'to find ways to represent continuous properties of the world by discrete systems of symbols'. Notice how similar this is to the programme enunciated by Hamblin nearly twenty years earlier!

I believe that philosophy and AI can combine fruitfully to explore good ways of representing our knowledge about space, time, and change. The two enterprises may, to be sure, differ in their ultimate goals: at any rate the goal of AI has an essential technological component in that it seeks to build software systems that can handle the human knowledge in an approximately human way. Philosophy does not have *this* as its goal, but on the other hand philosophical speculation about space, time, and change ranges from the metaphysical ('What are these things *really*?'), through the epistemological ('What is the nature of our knowledge about them?), to the logical ('How can we represent this knowledge systematically and unambiguously?), and the second and third of these are directly relevant to the technological goals of AI. My aim in this paper has been to use the history of a particular problem, the problem of the Instant of Change, to illustrate just how close are the interests of at least some philosophers and some AI researchers.

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