MSc Module MTMW14:
www.met.rdg.ac.uk/cag/courses

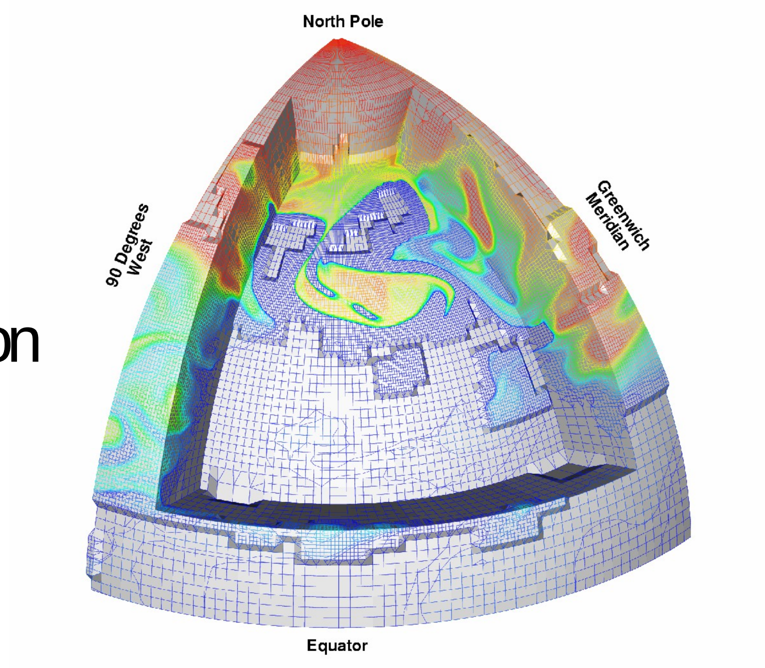
Numerical modelling of atmospheres and oceans

The Basics

1.1 Introduction

1.2 Brief history of numerical weather prediction

1.3 Dynamical equations for the unforced fluid



1.1 Introduction: Aims of this course

By the end of this module YOU should be able to:

- explain the main components that make up a numerical atmosphere/climate model
- recognise the strengths and weaknesses of the main numerical methods used to model the atmosphere and oceans.
- develop your own numerical models in Fortran 90.

Week 1: The Basics

1.1 Introduction

1.2 Brief history of numerical weather forecasting and climate modelling

1.3 Dynamical equations for the unforced fluid (the “dynamical core”)

Week 2: Modelling the real world

2.1 Physical parameterisations: horizontal mixing and convection

2.2 Ocean modelling

Week 3: Staggered schemes

3.1 Staggered time discretisation and the semi-implicit method

3.2 Staggered space discretisations

Week 4: More advanced spatial schemes

4.1 Lagrangian and semi-lagrangian schemes

4.2 Series expansion methods: finite element and spectral methods

Week 5: Synthesis

5.1 Revision

5.2 Test

1.1 Introduction: MTMW14 Timetable 2006

Week	Date	11am-1pm G02 Palmer	2-5pm Computer lab GL68
1	10 Jan	1.1 Introduction 1.2 Brief history 1.3 Dynamical equations	Computer assignment
2	17 Jan	2.1 Physical schemes 2.2 Ocean modelling	Computer assignment
3	24 Jan	3.1 Staggered time schemes 3.2 Staggered space schemes	Computer assignment
4	31 Jan	4.1 Lagrangian methods 4.2 Series methods	Computer assignment
5	7 Feb	5.1 Revision 5.2 Test	Computer assignment

Notes:

- deadline to hand in practical assignment: Friday of Week 10 Spring Term.

1.1 Introduction: Prerequisites and Assessment

- Some knowledge of fluid dynamics
- Some knowledge of numerical methods
- Some ability to use Fortran and MATLAB
- + plenty of curiosity !

- 5 week practical assignment [70 % total]
- 1 hour test in week 5 [30% total]

1.1 Introduction: Books on numerical modelling

1. McGuffie, K. and Henderson-Sellers, A. 1997 “A Climate Modelling Primer”, Wiley.

2. Kalnay, E. (2002) Atmospheric Modeling, Data Assimilation and Predictability, Cambridge University Press, 512 pages.

3. Washington, W.M, Parkinson, C.L. (1986) Introduction to Three Dimensional Climate Modelling, 422 pages.

4. Trenberth, K.E. (Editor) 1992 “Climate System Modeling”, Cambridge University Press

5. Haltiner, G.J. and Williams, R.T. 1980 “Numerical prediction and dynamical meteorology”, 2nd edition.

6. Durran, D.R. 1999 “Numerical methods for wave equations in geophysical fluid dynamics”, Springer.

1.1 Introduction: Ocean books and other sources

1. **Dale B. Haidvogel, Aike Beckmann (1999) Numerical Ocean Circulation Modeling, Imperial College Press, 300 pages.**

2. **Chassignet, E.P. and Verron, J. (editors) 1998 "Ocean modeling and parameterization", Kluwer Academic publishers**

3. **+ many articles in journals such as J. Climate, QJRMS, etc.**

1.1 Introduction: Internet sites

- **eumetcal.meteo.fr/article.php3?id_article=58**

  Very good *online* ***Numerical Weather Prediction*** *course*

- **www.ecmwf.int/resources/meteo-sites.html#members**

  List of *National Weather Services in Europe*

- **www-pcmdi.llnl.gov**

  Atmosphere Model Intercomparison Project

- **www.ecmwf.int/research/ifsdocs**

  Documentation for ECMWF forecasting model

1.1 Introduction: Why do we need models ?

- To ESTIMATE the state of the system:

  ANALYSES=OBSERVATIONS + MODEL

- To FORECAST the future state

- To SUMMARISE our understanding

  (MODEL=THEORY/MAP OF REALITY)

1.1 Introduction: What exactly is a "model" ?

model n. [Fr. Modele, It. Modello, from L. modellus]
A miniature representation (small measure) of a thing, with the several parts in due proportion.

- A model is only a "representation" of reality (e.g. a street plan of reality)
- Good modellers know the <u>strong</u> AND <u>weak</u> points of their models
- "Modelling" (English) and "Modeling" (American)
- Some quotations:

"All models are wrong, but some are useful" – George Box

"The purpose of models is not to fit the data but to sharpen the questions" – Samuel Karlin

"A theory has only the alternative of being right or wrong. A model has a third possibility, it may be right, but irrelevant." – Manfred Eigen

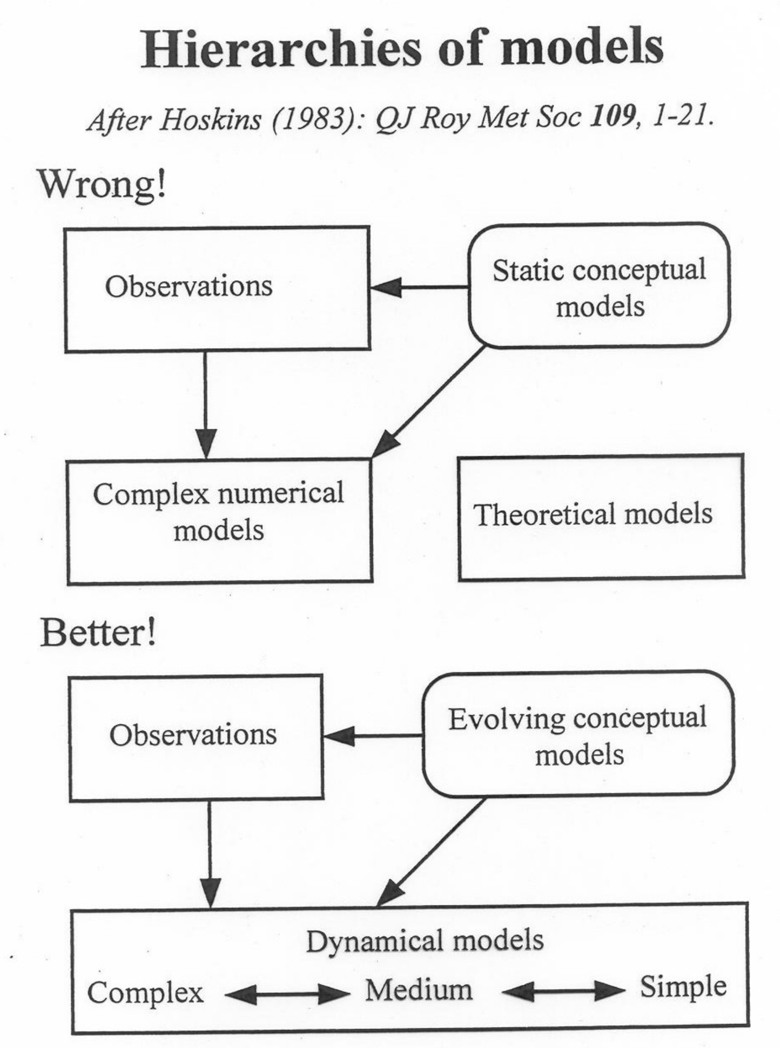
1.1 Introduction: The Modelling Process

1. COLLECT measurements
2. EXPLORE observed measurements
3. IDENTIFY a suitable class of models
4. ESTIMATE model parameters (tuning/fitting)
5. PREDICT new things using the model
6. EVALUATE performance
7. ITERATE – go back to previous steps

The Principle of Parsimony (Occam's razor)

Do not make more assumptions than the minimum needed.

→ Use the simplest model possible

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1.2 Brief history: quick overview

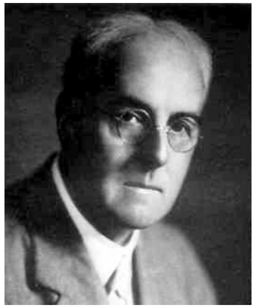
Year	Development
1643	Invention of the barometer by Torricelli
1820	First synoptic map made (using 50 year old data!)
1845	Invention of the telegraph
1850-	National Weather Services established – “empirical era”
1904	Vilhelm Bjerknes formulates the problem mathematically
1914-	L.F. Richardson tries to solve equations numerically
1950	Charney, Fjortoft, and Von Neumann make 24h forecast
1950s	3-dimensional numerical models developed at Princeton
1963	First numerical climate (general circulation) simulation
1967	Global ocean model developed by Cox and Bryan
1970s	Coupled models, spectral methods, and start of ECMWF
1980s	Semi-lagrangian methods, data assimilation, etc.
1990s	Massively parallel processors (MPPs), ensemble forecasts
2000s	Earth system modelling …

1.2 Brief history

‘primitive’ equations — 1904

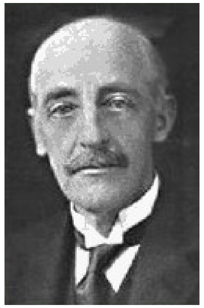
Vilhelm Bjerknes

manual forecast — 1922



Lewis Fry Richardson

southern oscillation — 1923



Gilbert Walker

computer forecasts — 1950

Jule G. Charney

40 Tflops 10 Tbytes — 2002

The Earth Simulator

1.2 Brief history: Svante August Arrhenius (1859-1927)

Concern of effect on climate of doubling CO_2

Arrhenius, S. 1896a
Uber den einfluss des atmosphaerischen kohlensauregehalts auf die temperatur der erdoberflaeche
Proc. of the Royal Swedish Academy of Sciences 22.

Arrhenius, S. 1896b
On the influence of carbonic acid in the air upon the temperature of the ground,
The London, Edinburgh, and
Dublin Philosophical
Magazine and Journal of Science, 41, 237-76.

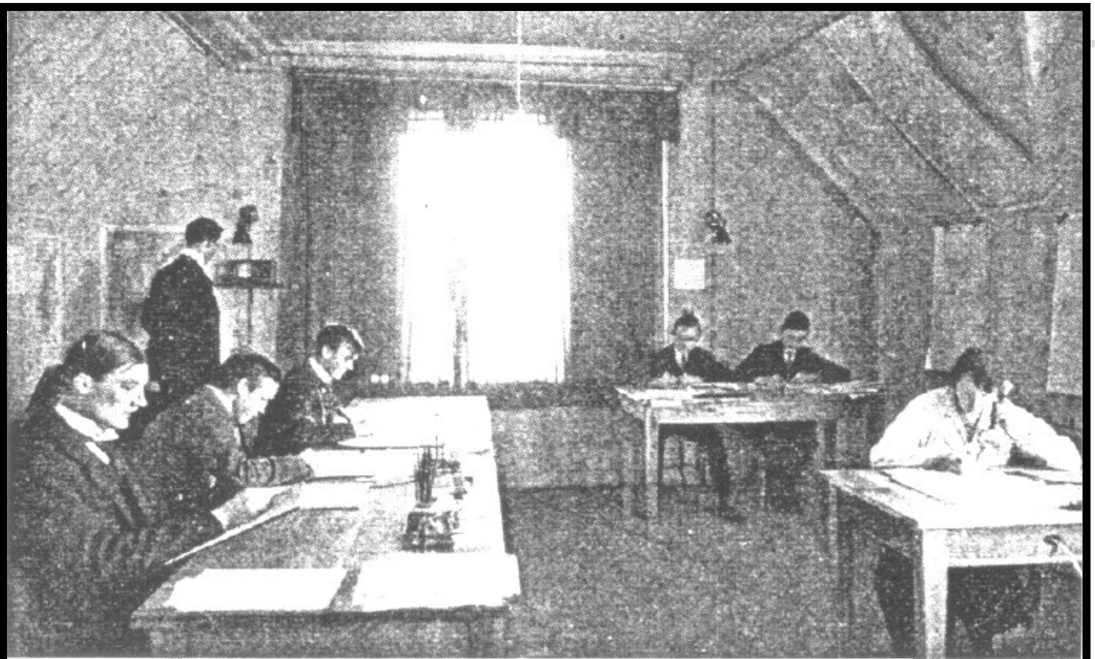
1.2 Brief history: Vilhelm Bjerknes (1862-1951)

Proposed weather forecasting as a deterministic initial value problem based on the laws of physics

V. Bjerknes, 1904:
Das problem von der wettervonhersage, betrachtet vom standpunkt der mechanik und der physik, Meteorologische Zeitschrift, Wien 21:1-7.

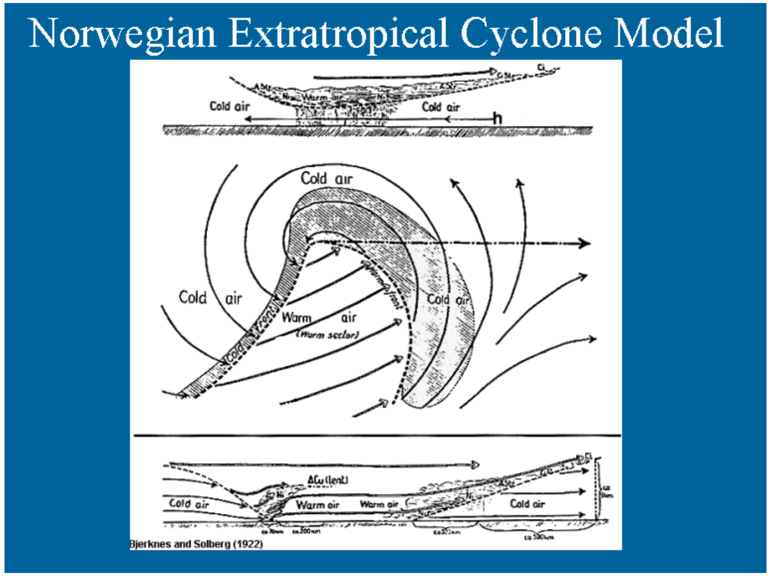
The problem of weather forecasting
as a problem in mechanics and physics

1.2 Brief history: The Bergen school



The Bergen Weather Service 14 Nov 1914

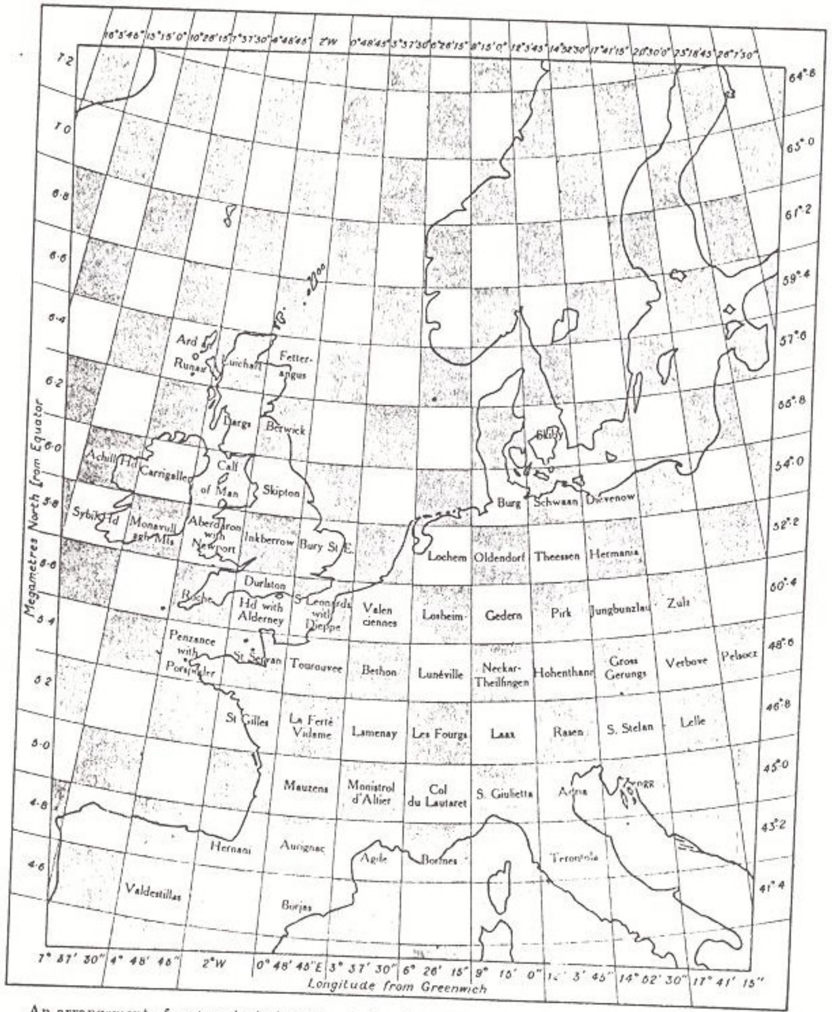
An attic-room, Allégaten 33, formerly a wealthy man's house, bestowed on Meteorology in 1918. — At the barograph: J. BJERKNES, chief. — At the left hand table (from left to right): T. BERGERON, C.-G. ROSSBY, S. ROSSELAND, junior meteorologists. — At the other tables: The technical staff



1.2 Brief history: Lewis Fry Richardson

- Lewis Richardson first attempted a numerical solution of atmospheric flow solutions – by hand !!
- The resulting prediction was highly unrealistic (due to CFL problems!)

Lewis Fry Richardson **(1881-1953)**



An arrangement of meteorological stations designed to fit with the chief mechanical properties of the atmosphere. Other considerations have been here disregarded. Pressure to be observed at the centre of each shaded chequer, velocity at the centre of each white chequer. The numerical coordinates refer to these centres as also do the names, although as to the latter there may be errors of 5 or 10 km. The word "with" in "St Leonards with Dieppe" etc. is intended to suggest an interpolation between observations made at the two places. See page 9, and Chapters 3 and 7. Contrast the existing arrangement shown on p. 184.

WEATHER PREDICTION

BY

NUMERICAL PROCESS

BY

LEWIS F. RICHARDSON, B.A., F.R.MET.SOC., F.INST.P.

FORMERLY SUPERINTENDENT OF ESKDALEMUIR OBSERVATORY

LECTURER ON PHYSICS AT WESTMINSTER TRAINING COLLEGE

$\Delta t = 12, 24h$

$\Delta p = 150 mb$!

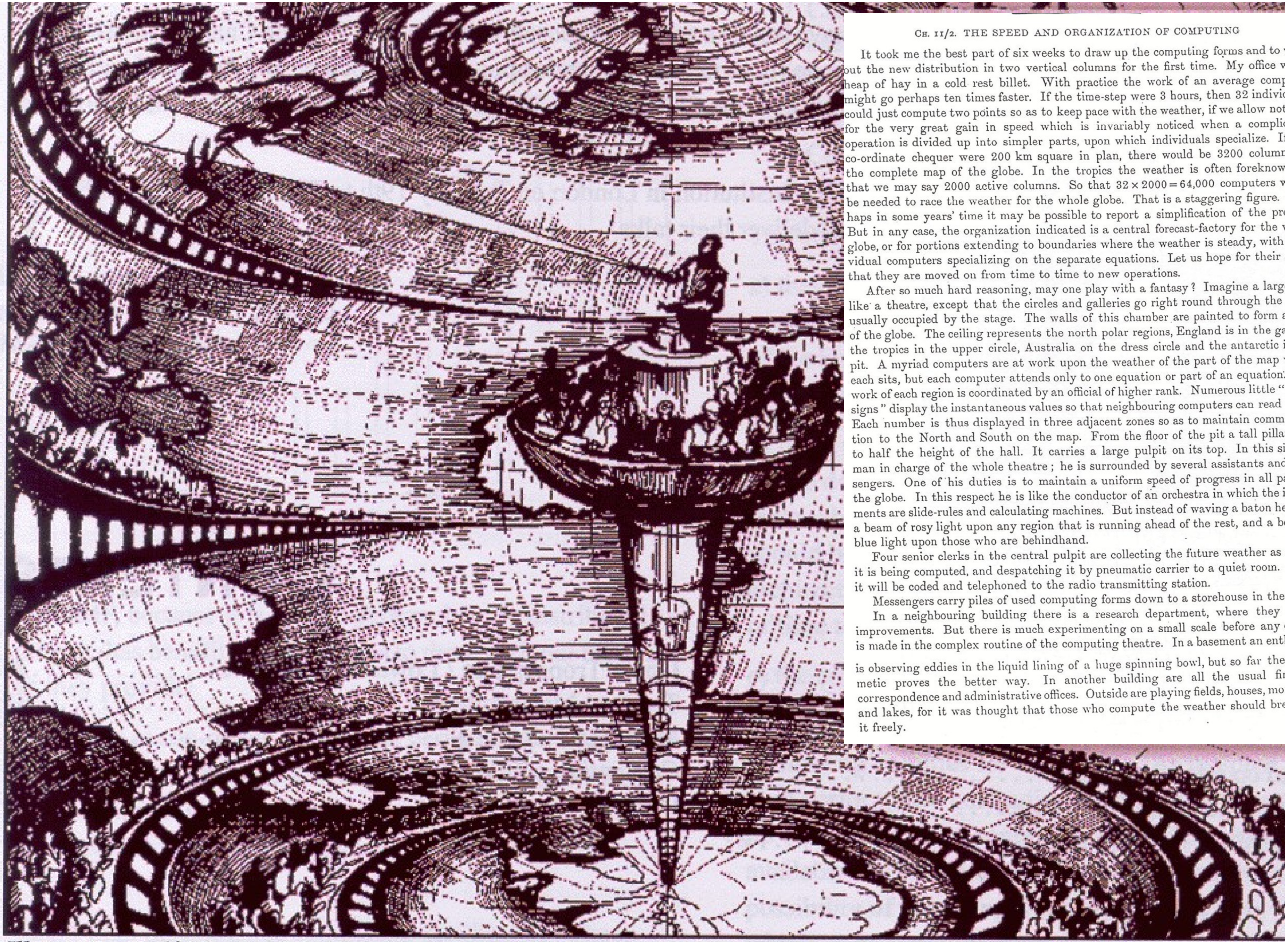
numerically unstable !

L.F. Richardson (1922)
Weather Prediction by Numerical Process,
Cambridge University Press

CAMBRIDGE

AT THE UNIVERSITY PRESS

1922



CH. II/2. THE SPEED AND ORGANIZATION OF COMPUTING

It took me the best part of six weeks to draw up the computing forms and to
out the new distribution in two vertical columns for the first time. My office v
heap of hay in a cold rest billet. With practice the work of an average comp
might go perhaps ten times faster. If the time-step were 3 hours, then 32 indivi
could just compute two points so as to keep pace with the weather, if we allow not
for the very great gain in speed which is invariably noticed when a complic
operation is divided up into simpler parts, upon which individuals specialize. I
co-ordinate chequer were 200 km square in plan, there would be 3200 column
the complete map of the globe. In the tropics the weather is often foreknow
that we may say 2000 active columns. So that $32 \times 2000 = 64{,}000$ computers v
be needed to race the weather for the whole globe. That is a staggering figure.
haps in some years' time it may be possible to report a simplification of the pr
But in any case, the organization indicated is a central forecast-factory for the
globe, or for portions extending to boundaries where the weather is steady, with
vidual computers specializing on the separate equations. Let us hope for their
that they are moved on from time to time to new operations.

After so much hard reasoning, may one play with a fantasy? Imagine a larg
like a theatre, except that the circles and galleries go right round through the
usually occupied by the stage. The walls of this chamber are painted to form a
of the globe. The ceiling represents the north polar regions, England is in the ga
the tropics in the upper circle, Australia on the dress circle and the antarctic i
pit. A myriad computers are at work upon the weather of the part of the map
each sits, but each computer attends only to one equation or part of an equation.
work of each region is coordinated by an official of higher rank. Numerous little "
signs" display the instantaneous values so that neighbouring computers can read
Each number is thus displayed in three adjacent zones so as to maintain comm
tion to the North and South on the map. From the floor of the pit a tall pilla
to half the height of the hall. It carries a large pulpit on its top. In this si
man in charge of the whole theatre; he is surrounded by several assistants and
sengers. One of his duties is to maintain a uniform speed of progress in all pa
the globe. In this respect he is like the conductor of an orchestra in which the i
ments are slide-rules and calculating machines. But instead of waving a baton he
a beam of rosy light upon any region that is running ahead of the rest, and a b
blue light upon those who are behindhand.

Four senior clerks in the central pulpit are collecting the future weather as
it is being computed, and despatching it by pneumatic carrier to a quiet room.
it will be coded and telephoned to the radio transmitting station.

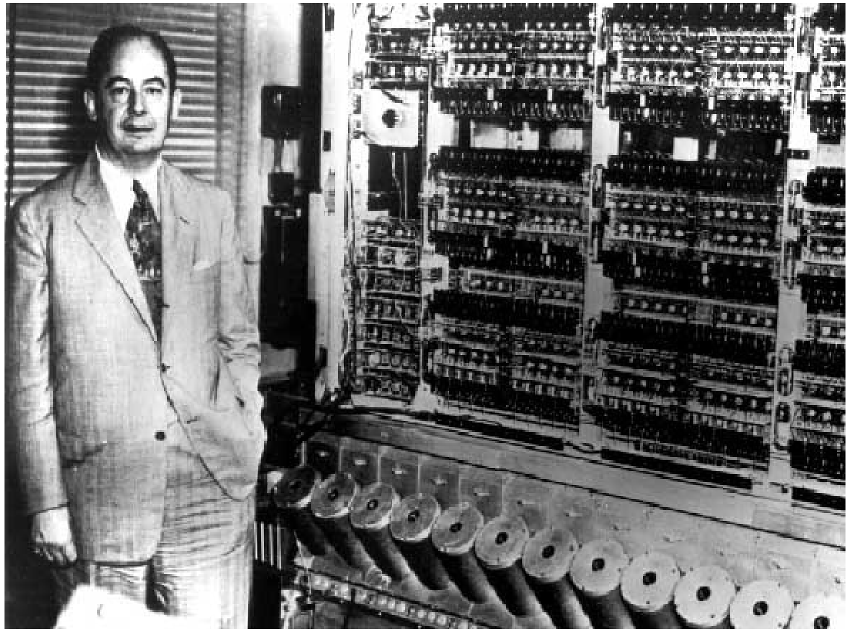
Messengers carry piles of used computing forms down to a storehouse in the

In a neighbouring building there is a research department, where they
improvements. But there is much experimenting on a small scale before any
is made in the complex routine of the computing theatre. In a basement an entl
is observing eddies in the liquid lining of a huge spinning bowl, but so far the
metic proves the better way. In another building are all the usual fi
correspondence and administrative offices. Outside are playing fields, houses, mo
and lakes, for it was thought that those who compute the weather should bre
it freely.

Illustration: Alf Lannerbäck, Sweden.

1.2 Brief history: The first electronic computer: ENIAC

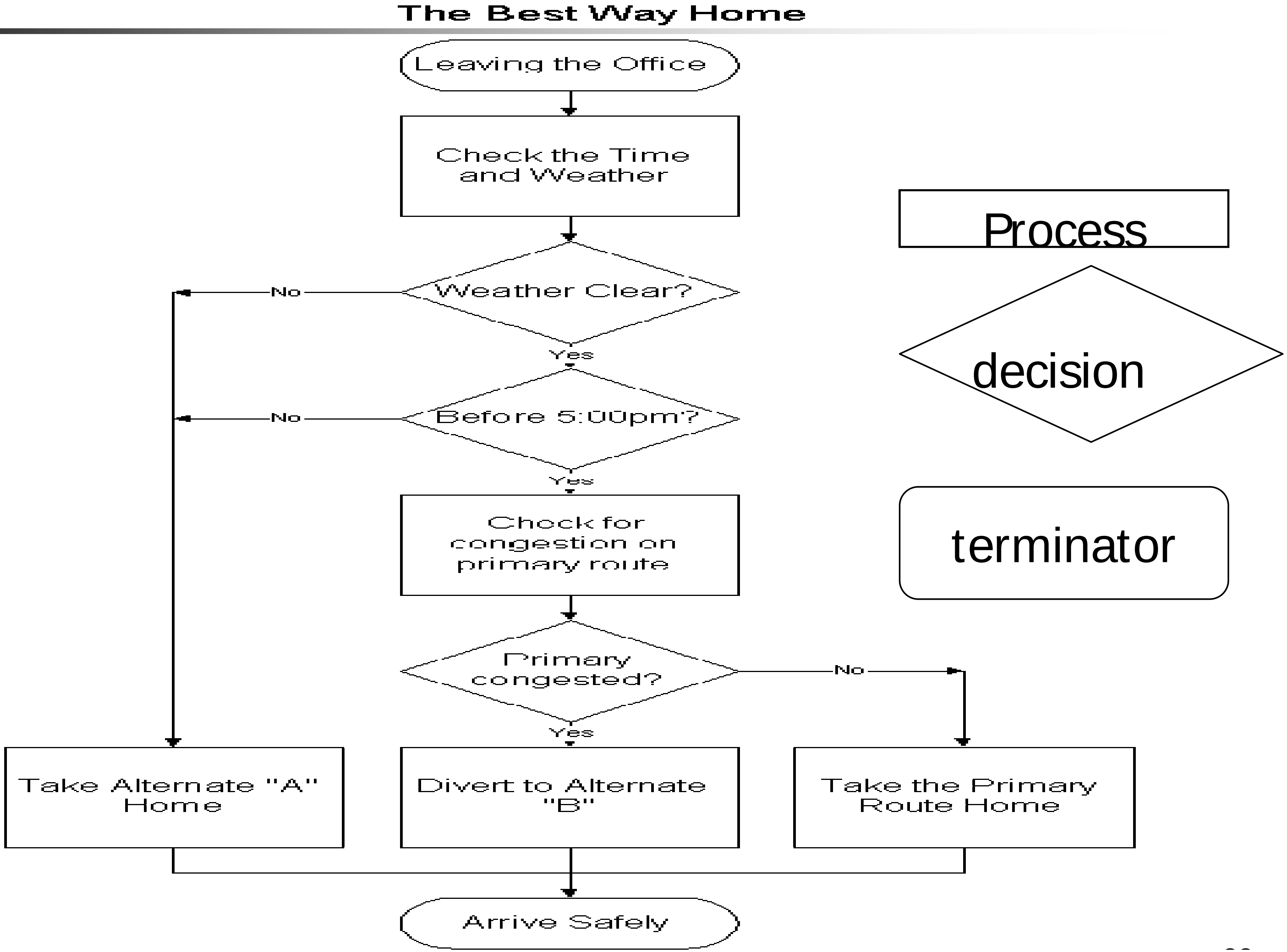
1946 - Studies of digital computers for the purpose of weather prediction were initiated by John Von Neumann.



1950 – Von Neumann, Charney, and Fjortoft led scientists in producing a retrospective 24hr forecast, using the ENIAC (Electronic numerical integrator and calculator)

The first numerical predictions in real time were prepared by C.G Rossbys' team at the International Meteorological Institute in Sweden.

1.2 Algorithms: How to do flow charts …



http://deming.eng.clemson.edu/pub/tutorials/qctools/flowm.htm

1.2 Brief history: Jule Gregory Charney (1917-1981)

Led the team that made the first numerical weather forecast on an electronic computer…

And played a key role in helping establish the satellite observing system

Charney, Fjörtoft, von Neumann
"Num. integration of the BVE"
Tellus 2 237-254 1950

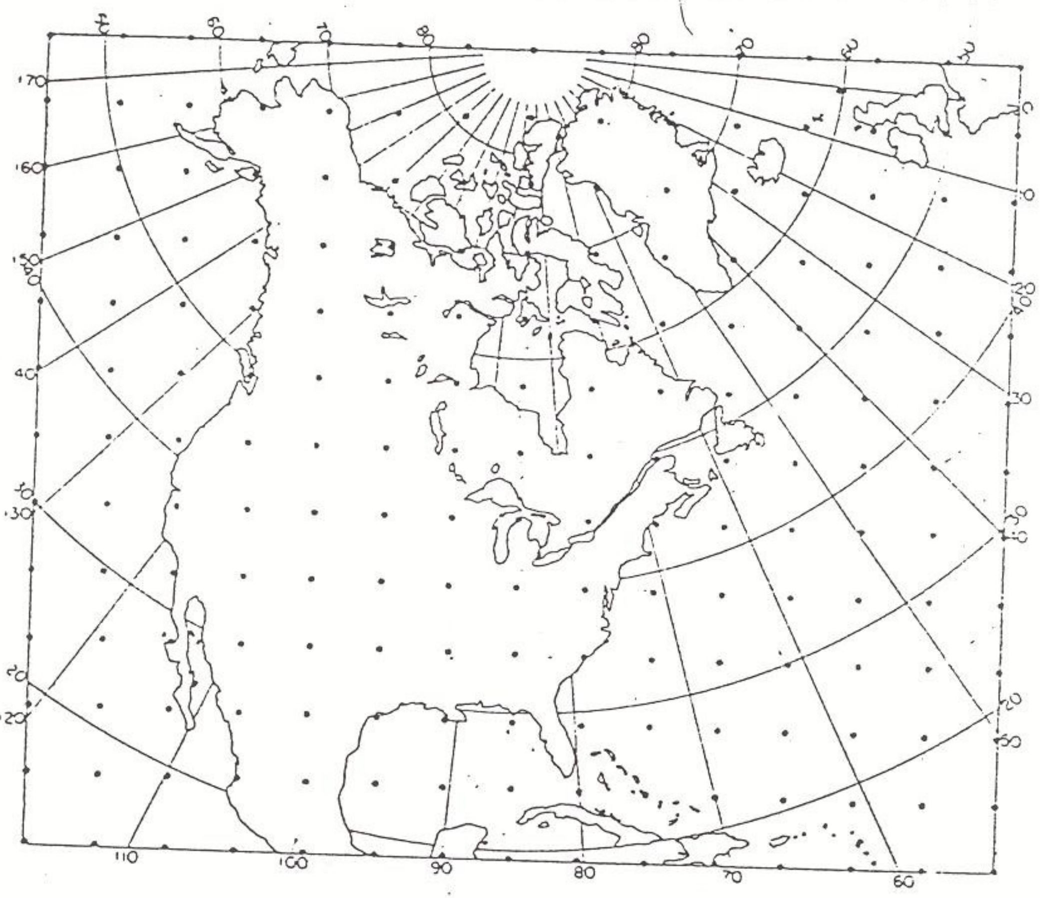
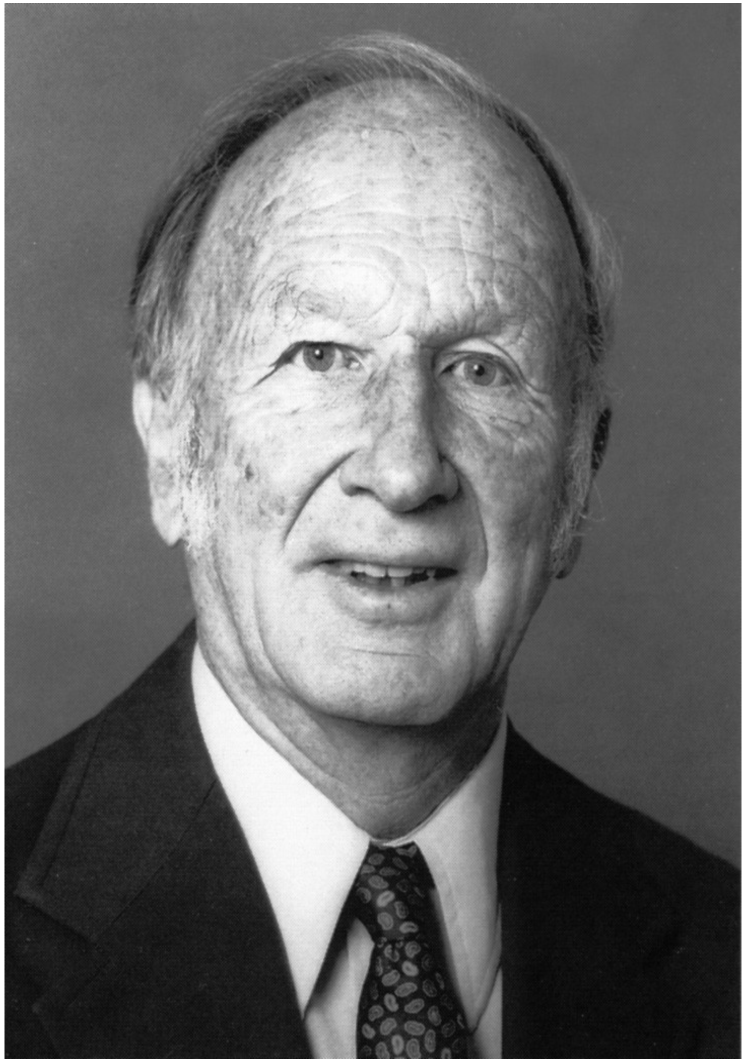


Fig. 1. A typical finite-difference grid used in the computations. A strip two grid intervals in width at the top and side borders and one grid interval in width at the lower border is not shown.

- ENIAC computer
- 15x18 grid points
- Resolution DX=736km
- Timestep h=1,2,3 hrs
- careful with numerics
- showed that numerical weather prediction was feasible.

1.2 Brief history: Edward Lorenz (1917 –)

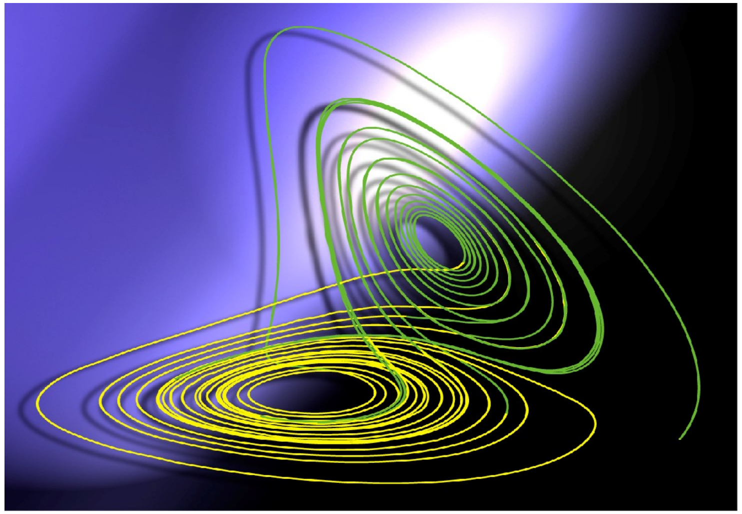


"… one flap of a sea-gull's wing may forever change the future course of the weather" (Lorenz, 1963)

$$\dot{X} = -\sigma X + \sigma Y$$

$$\dot{Y} = -XZ + rX - Y$$

$$\dot{Z} = XY - bZ$$



1.2 Brief history: Syukuro Manabe (1931-)

First predictions of global warming based on comprehensive global climate models

Smagorinsky, J., S. Manabe, J. L. Holloway, Jr., 1965:
Numerical results from a nine-level general circulation model of the atmosphere.
Monthly Weather Review, 93(12), 727-768.

Manabe, S., and R. T. Wetherald, 1975:
The effects of doubling CO2 concentration on the climate of a general circulation model.
Journal of the Atmospheric Sciences, 32(1), 3-15.

Manabe, S., K. Bryan, and M. J. Spelman, 1975:
A global ocean-atmosphere climate model.
Part I. The atmospheric circulation.
Journal of Physical Oceanography, 5(1), 3-29.

1.2 Brief history: The Met Office and its Computers

1981 The met offices first super computer, the CDC Cyber205 was installed.

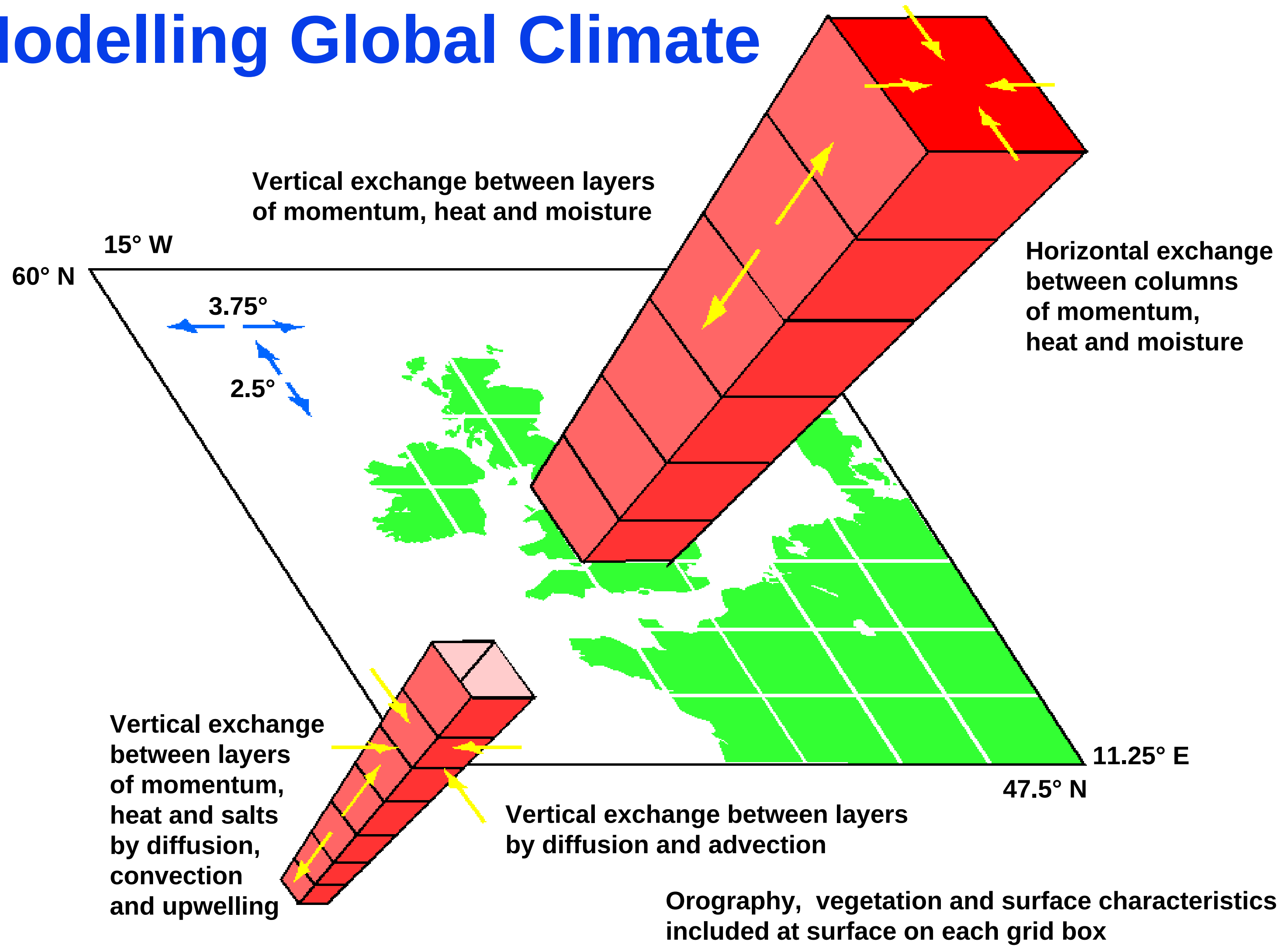
1990- Replaced by a Cray Y-MP.

1996 – The Cray T3E super computer was installed

1999 - A 2nd T3E was installed.

The biggest super computer in the world produces forecasts for the Japanese Meteorological Agency.

Modelling Global Climate



From: Hack, J.J., 1992: Chapter 9 of
Climate System Modelling, Editor: K.E. Trenberth, CUP.

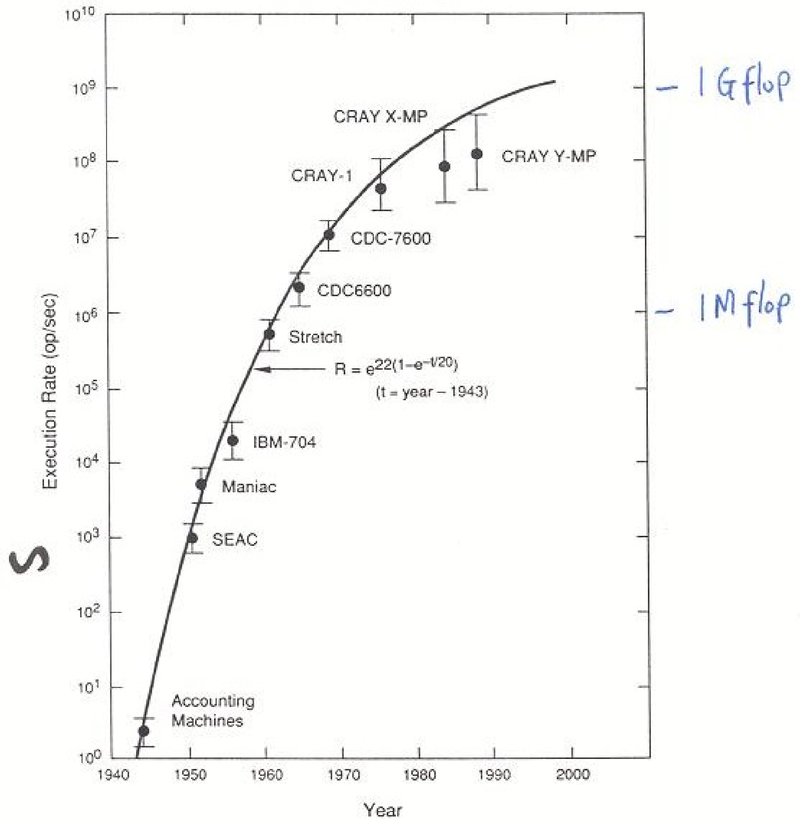


Fig. 9.1 Trend in single processor computational performance (after Worlton, 1987 [personal communication]).

(e.g., FORTRAN) began to appear along with multiprogrammed operating systems, both of which helped to reduce the complexity of implementing application programs. The later introduction of integrated circuit technology contributed to improved performance along with higher reliability, while enabling innovative improvements in machine organization, i.e., the implementation of the central processor architecture.

The late 1960s and early 1970s saw the introduction of vector extensions to the von Neumann architecture, further improving processor utilization. Two different approaches emerged during this period which

S doubles about every 18 months!

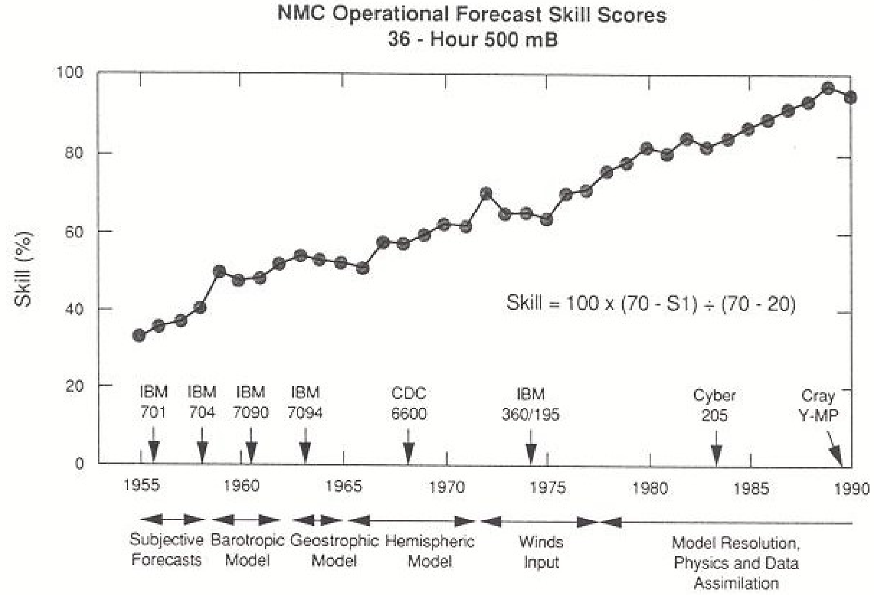
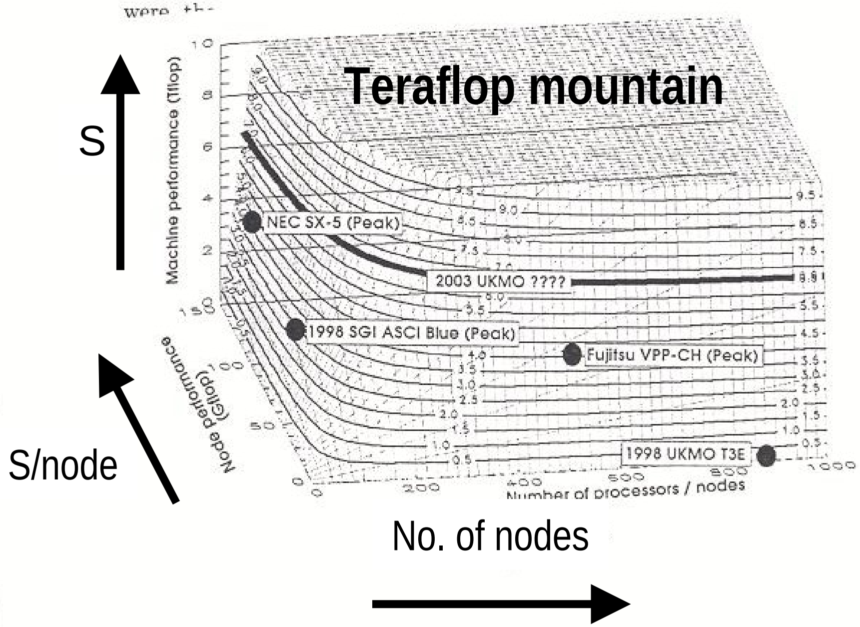


Fig. 9.2 36 hour 500 mb forecast skill over North America at the National Meteorological Center (after Kálnay et al., 1991).



Earth System Simulator - Japan

UGAMP newsletter, 26, Oct 2002

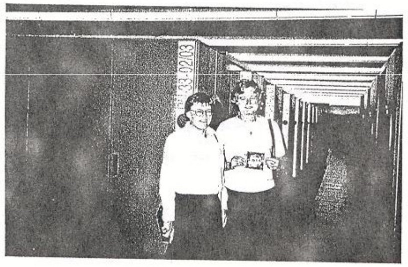
Bigger, Faster, Better HPC Facilities?

Lois Steenman-Clark (lois@met.rdg.ac.uk), CGAM, Dept. of Meteorology, University of Reading.

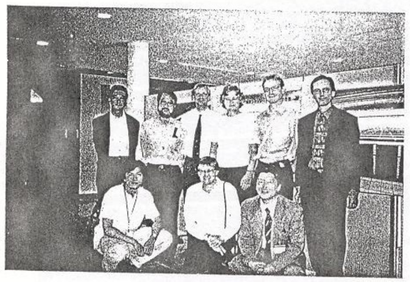
The Top 500 list of the World's high performance computers is like a computer version of the Japanese Sumo wrestling league, but where strength and speed are measured bi-annually by an idealised benchmark, Linpack. Number 1 in this list, the Earth Simulator in Japan, an NEC machine with a peak speed of 40 Tflops, tops the Top 500 list by almost a factor 5. The performance of the Earth Simulator equals the sum of the performance of the next 12 computers in the list.

Table 1: Extract from the Top 500 list of the world's fastest computers

Rank	Manufacturer	Computer	Linpack (Gflops)	Country	Year	Processors
1	NEC	Earth-Simulator	35860	Japan	2002	5120
2	IBM	ASCI White, SP Power3 375Mhz	7226	USA	2000	8192
3	Hewlett-Packard	Alphaserver SC ES45/1Ghz	4463	USA	2001	3016



Above: Professor Julia Slingo and Dr. Lois Steenman-Clark (CGAM) in the Earth Simulator machine room. Julia was presented with a CD-ROM of the first results from a very high resolution climate model run on the Earth Simulator. Below: Julia and Lois with David Griggs, Malcolm Roberts, Richard Woods from the Hadley Centre together with Dr. Sato (front right), Director of the Earth Simulator and Professors Sumi and Kimoto (back).



Alan Thorpe visited the Earth Simulator in December last year and participated in discussions of possible UK-Japan collaborative projects which would enable UK high resolution climate models to be run on the Earth Simulator. The Earth Simulator began operation in March 2002 and discussions have been continuing. To further these discussions Julia Slingo and I, together with David Griggs, Director of the Hadley Centre and Richard Woods and Malcolm Roberts, both from the Hadley Centre, visited Japan in September 2002, firstly for a 2 day scientific workshop at the University of Tokyo and then to visit the Earth Simulator in Yokohama.

The Earth Simulator machine room is the size of a very large sports hall and very impressive (see photograph). Given the size of the machine perhaps the most impressive thing was the fact that most of the little green lights on top of each of the 640 nodes (each node has 8 processors), which lit up when the node was active, were on. So after 6 months in operation the machine is already fairly full and doing science. We were told about the stringent performance tests the models need to pass before they can use the Earth Simulator and shown the first results from their 10 km resolution global atmosphere and ocean simulations. News of possible CGAM and other projects on the Earth Simulator will be available soon.

The recent announcement that the Met. Office will replace their Cray T3Es with an NEC SX6 when they move to Exeter next year is fortuitous, as the NEC SX6 is a very similar architecture to the Earth Simulator. The Met Office are already planning porting the Unified Model to the NEC. The new Met Office NEC computer will initially have a peak speed of 2 Tflops.

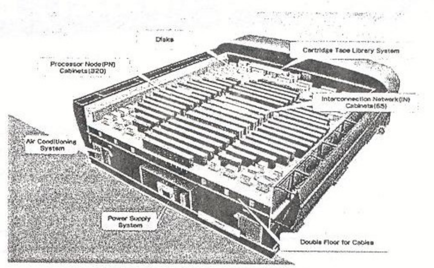
We should compare the speed of these NEC machines to that of the machine procured by the UK Research Councils for the new capability service (HPCX). This new machine will be an IBM, initially with 1280 processors and a peak speed of nearly 7 Tflops. There are plans for significant enhancements after 2 and then 4 years. This machine will be housed at Daresbury, and the service will be run jointly by Edinburgh and Daresbury. The HPCX service will start at the end of this year, although there will be some early-user projects, including CGAM, running before then. Details of the new HPCX capability service are now available on the service web site (http://www.hpcx.ac.uk). The installation of this new machine will put the UK in the top league of the Top 500 list. ECMWF have also announced that the replacement of their Fujitsu will be a similar IBM system. We look forward to collaborating with ECMWF on many technical issues.

For the HPCX IBM machine the definition of capability is flexible but initially it is expected to describe jobs which require over 512 processors, that is 40% of the total machine. It is a challenging prospect scaling our current simulations up to this size. The challenge is not just scientific but also technical. The arrival of the IBM machine will be the first tightly-linked cluster of shared memory nodes we had access to in the UK. CGAM plan to use the HPCX early-user project to gain technical experience of cluster computing and to explore the issues of increasing the spatial resolution of the Unified Model, all of which will be useful for projects for the Earth Simulator.

If we compare the peak speeds of the machines accessible to UK climate scientists to the Earth Simulator then we are currently a long way behind in terms of capability (see figure right). Even allowing for some inaccuracies in my interpretation of some of the longer term plans, after 5 years the Earth Simulator will still be bigger than any machine in the UK by a significant factor.

Bigger is not always better, as not all modelling experiments need capability, some need capacity i.e. good throughput. In terms of capacity CSAR will still continue to provide most of this service nationally. At CSAR the SGI Origin systems are being expanded and they are more closely linked with the storage area network (SAN). The proposal by NERC to close the CSAR Fujitsu service at the end of this year will severely affect UGAMP but plans for this closure are still not complete and I hope to have more news very shortly.

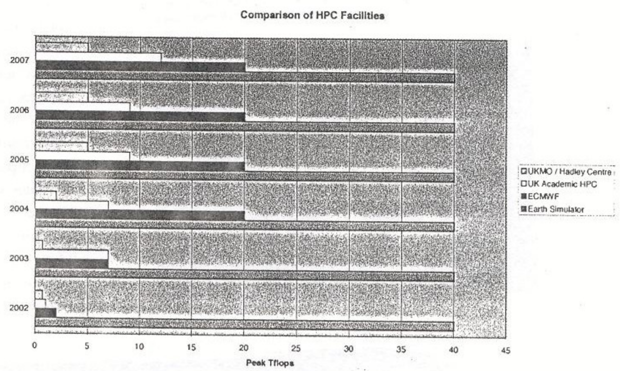
The management of change, whether it is changes in computer architecture or HPC service provision or models, within a programme such as UGAMP is never easy. As information is key to managing change, the new CGAM web service, which will soon be part of the NCAS web site, is being planned to provide a much improved information service. If there is any particular information that CGAM or you can provide that would enhance the information service for the UGAMP community, then please contact me.



Earth Simulator Specifications

- 640 processor nodes.
- Each node consists of eight vector processors connected via a high speed network.
- Peak performance per processor: 8 Gflops.
- Peak performance per node: 64 Gflops.
- Shared memory per node: 16Gbytes
- Total peak performance: 40 Teraflops
- Total main memory: 10 Tbytes

For more details, visit the Earth Simulator website: http://www.es.jamstec.go.jp/



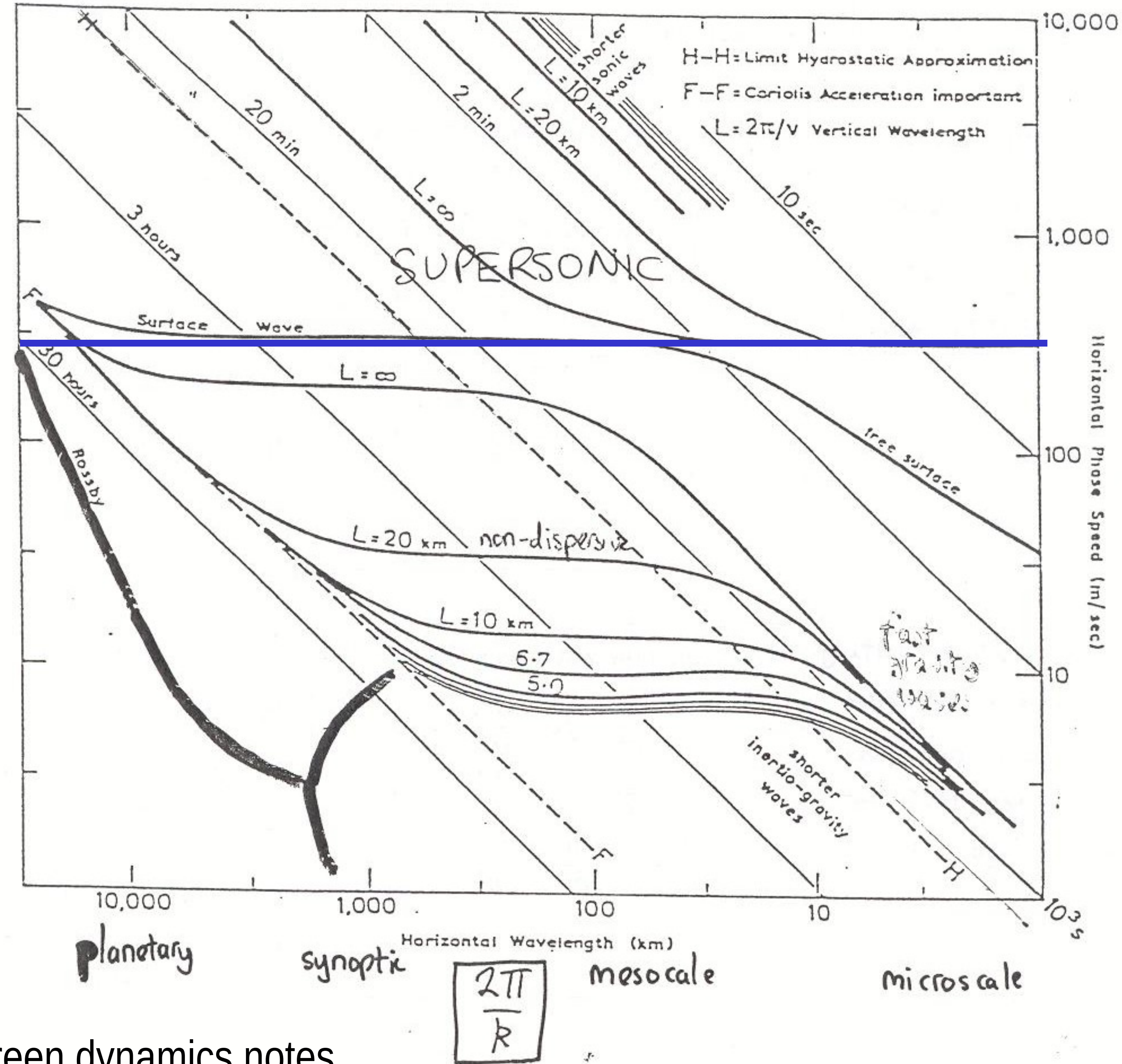
1.2 Summary of driving factors

- Forecasting needs:
  - *Nowcasting (up to 6 hours lead)*
  - *Short-range (6h-3days)*
  - *Medium-range (3-10 days)*
  - *Extended-range (10-30 days)*
  - *Long-range (>30 days) – seasonal/climate forecasts*

- Computer technology – speed and memory

- Availability of more data e.g. satellite measurements

1.3 Waves in a compressible atmosphere



From J.S.A. Green dynamics notes

1.3 Dynamical equations: hierarchies of models

- **Molecular dynamics** – predict the motion of 10^{45} molecules using Newton's laws (not feasible!)
- **Navier-Stokes equations** – use the continuum approximation to treat gas as a continuous fluid. Includes sound, gravity, inertial, and Rossby waves.
- **Euler equations** – use the incompressible or anelastic approximations to filter out sound waves that have little relevance to meteorology (speeds > 300m/s)
- **Primitive equations** – use the hydrostatic approximation to filter out vertically propagating gravity waves (speeds >30m/s)
- **Shallow water equations** – linearise about a basic steady flow in order to describe horizontal flow for different vertical modes. Each mode has a different equivalent depth external mode (e.g. tsunami) + barotropic mode + baroclinic modes
- **Vorticity equations** – assume geostrophic balance to remove gravity waves. Useful in understanding extra-tropical dynamics.
- **More conceptual models**: 0d (low order), 1d, and 2d models.

1.3 Some basic sets of equations

Primitive equations

$$\frac{Du}{Dt} + fz \times u + \nabla_p \Phi = 0$$

$$\frac{\partial \Phi}{\partial p} + \frac{RT}{p} = 0$$

$$\nabla_p . u + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{D \log T}{Dt} - \frac{\kappa \omega}{p} = 0$$

Shallow water eqns

$$\frac{\partial u}{\partial t} + fv + \frac{\partial \Phi}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - fu + \frac{\partial \Phi}{\partial y} = 0$$

$$\frac{\partial \Phi}{\partial t} + gH\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Barotropic vor. eqn

$$\frac{D\xi}{Dt} + \beta v = 0$$

Exercise:

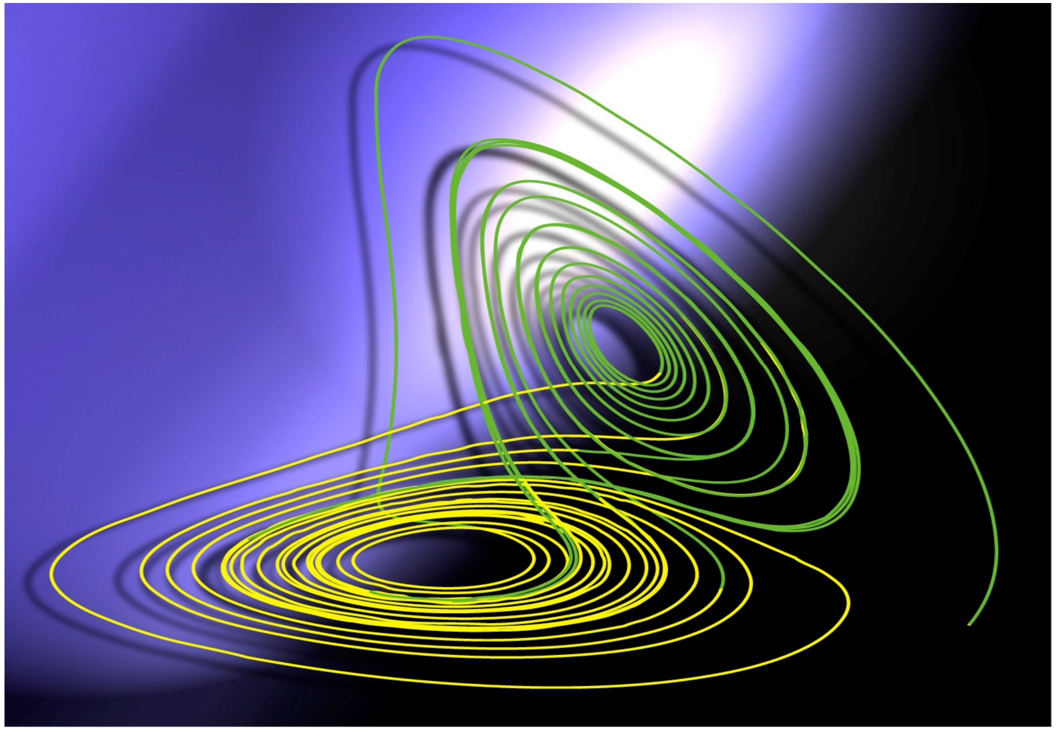
Derive prognostic equations for relative vorticity and divergence using the shallow water equations and compare the resulting vorticity equation to the barotropic vorticity equation.

1.3 Dynamical systems theory

$$\frac{\partial\Psi}{\partial t} + Q[\Psi] = F[\Psi; t, \theta]$$

Ψ is a multidimensional *state vector*

Q is hydrodynamical advection term (*dynamical core*)
F is forcing and dissipation due to physical processes



The state of the system can be represented by a point in *state space*.

Movement of point (flow) in state space represents the evolution of the system

1.3 Why do we use numerical approximation?

Can't represent an infinite dimensional state space on a finite memory computer → discretise or truncate horizonal/vertical space to a finite set of points/modes so that **partial differential equations** become a **finite set of ordinary differential equations**.

Can't represent continuous time on a finite memory computer → discretise flow through state space into a discrete set of hops (time discretisation) so that **ordinary differential equations** become a set of **finite difference equations**.

$$\Psi^{(n+1)} = M(\Psi^{(n)})$$

e.g. the logistic map:

$$z^{(n+1)} = \gamma z^{(n)}(1 - z^{(n)})$$

1.3 Finite difference methods in time & space

Idea: Replace all derivatives by finite difference approximations:

For example: $$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$ 1-dimensional advection equation

$$\phi_j^{(n)} = \phi(x_j, t_n) = \phi(j\Delta, nh)$$

$$\frac{(\varphi_j^{(n+1)} - \varphi_j^{(n-1)})}{2h} + u_i^n \frac{(\varphi_{j+1}^{(n)} - \varphi_{j-1}^{(n)})}{2\Delta} = 0$$

Centered Time Centred Space (CTCS)

n= "time level" j= "grid point"

h= "time step" delta= "grid spacing"

1.3 Stability, accuracy, and convergence

- **Stability** – numerical solution remains bounded

$$\phi^{(n)} \text{ stays finite as } n \to \infty$$

Types: zero, absolute, conditional

Methods: Courant-Friedrichs-Lewy, Von Neumann, Energy, …

- **Accuracy** – closeness of numerical solution to true solution

$$\text{"local error"} = \phi(\mathrm{t}_\mathrm{n}) - \phi^{(n)}$$

$$\text{"truncation error"} = \mathrm{FDE}(\phi) = O(h^p)$$

- **Convergence** – how the numerical solution converges to

$$\| \phi(t_n) - \phi^n \| \sim O(h^q) \text{ in limit as } h \to 0$$

1.3 Numerical instability

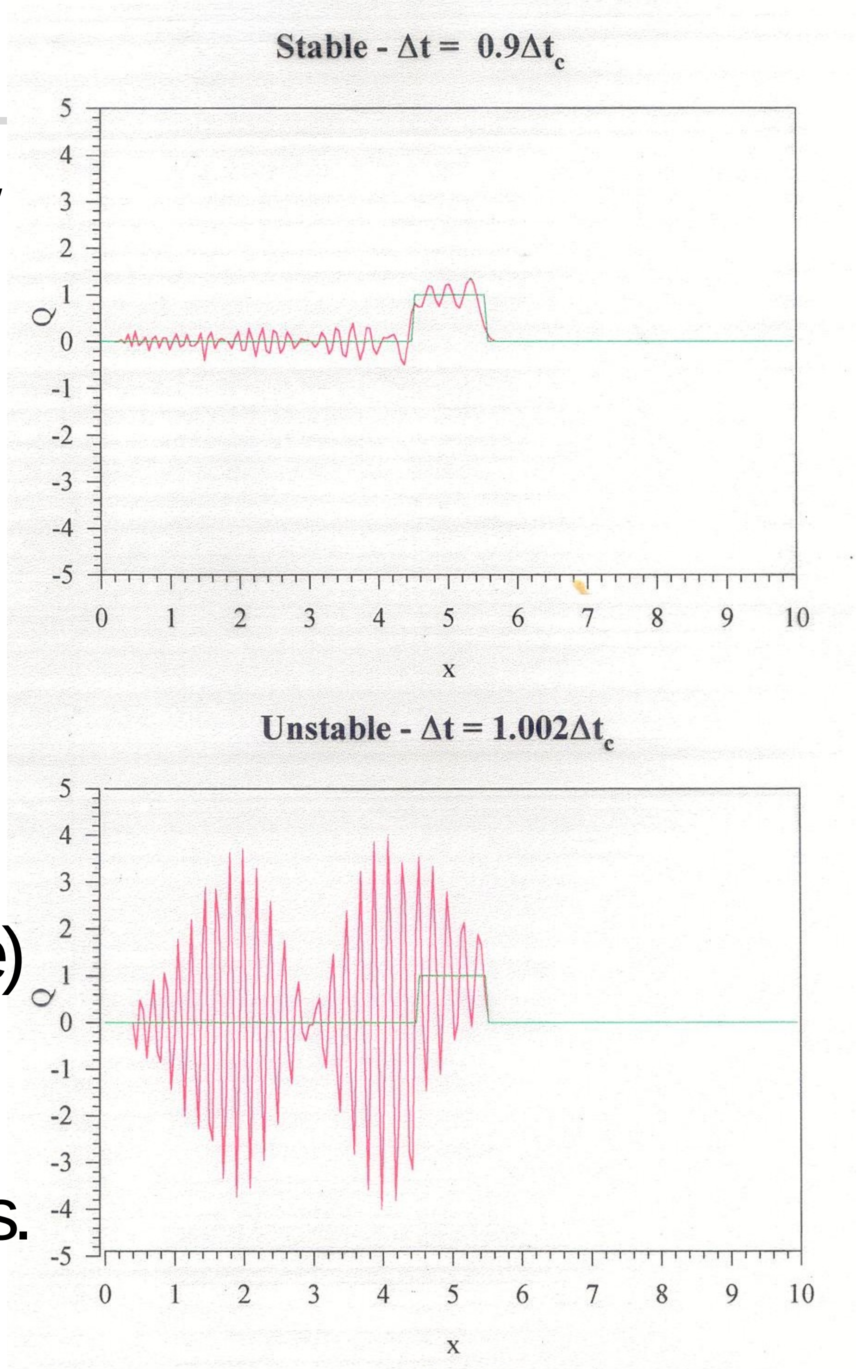
Numerical solutions that grow exponentially with time step

Lax equivalence theorem:

A stable p'th order accurate linear finite difference equation is p'th order convergent

So stable & *consistent* (p>=1 accurate) linear schemes converge on the true solution as h→0.

E.g. stable 2nd order leapfrog schemes.



1.3 Courant-Friedrichs-Lewy stability analysis (CFL)

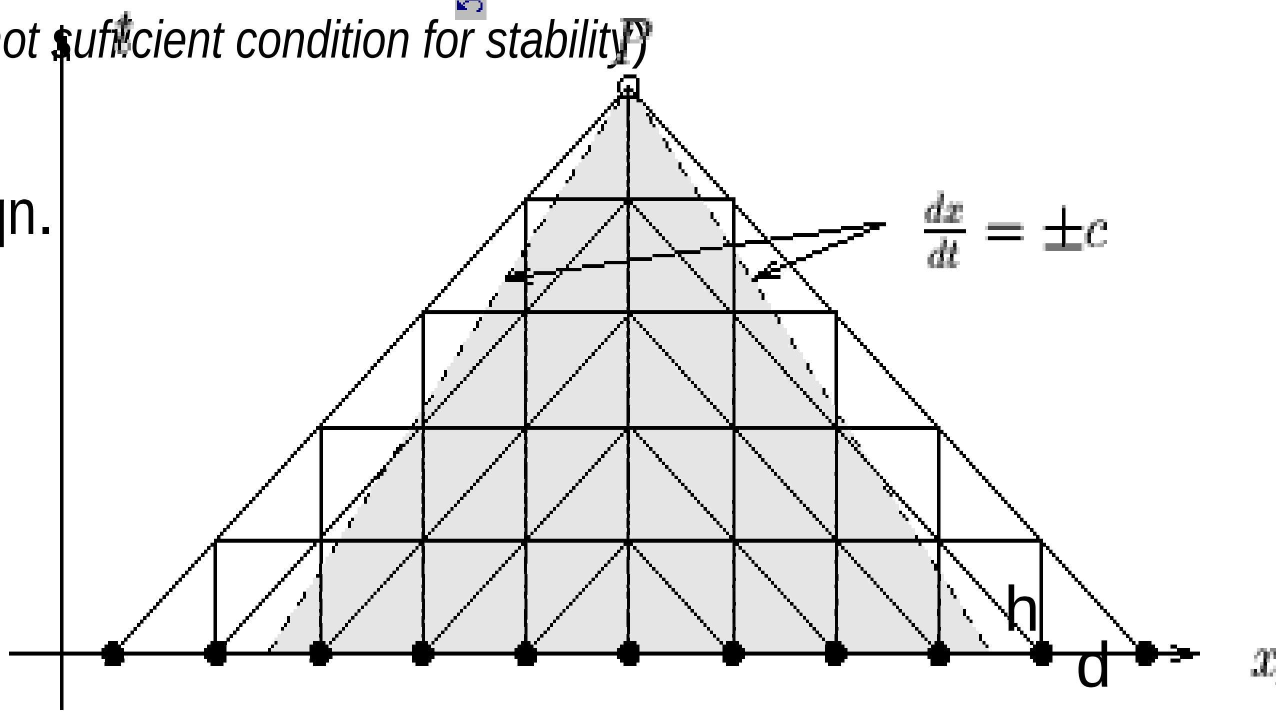
R. Courant, K.O. Friedrichs, and H. Lewy, 1928: Uber die partiellen Differenzengleichungen der mathematischen Physik. Mathematische Annalen, Vol. 100, pages 32-74. http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D29345&p=36

Domain of dependence of the finite difference scheme must include the domain of dependence of the partial differential equation.

(*necessary but not sufficient condition for stability*)

Example:

FDE of advection eqn. where the value on the next time step depends on nearest neigbour values.



1.3 Implications of CFL

- Information should not spread more than one grid spacing in one time step (for a FTCS advection scheme!)

- Time step needs to be small enough to avoid instability caused by the fastest waves e.g. $h<d/c$

  → fast waves require lots of time steps & computer time!

  → filter out fastest waves, slow down waves, move grid with the flow

- Note that c is the “phase velocity” simulated by the finite difference equations and so methods that can slow down unwanted fast waves can be used to circumvent CFL.

- Diffusion and advection can also cause CFL instability!

1.3 Von Neumann stability analysis

Idea: Consider spatial Fourier solutions of form $\phi_j^n = A^n e^{ik(j\Delta)}$

For example: Stability of BTCS scheme for1d advection equation

$$\frac{(\varphi_j^{(n+1)} - \varphi_j^{(n)})}{h} + u\frac{(\varphi_{j+1}^{(n+1)} - \varphi_{j-1}^{(n+1)})}{2\Delta} = 0$$

$$\Downarrow$$

$$\left(\frac{(A-1)}{h} + u\frac{(e^{ik\Delta} - e^{-ik\Delta})A}{2\Delta}\right) A^n e^{ik(j\Delta)} = 0$$

1.3 Von Neumann stability analysis

$$\left(\frac{(A-1)}{h} + u \frac{(e^{ik\Delta} - e^{-ik\Delta})A}{2\Delta} \right) A^n e^{ik(j\Delta)} = 0$$

$$\Rightarrow \quad A = \left(1 + \frac{iuh}{\Delta} \sin k\Delta \right)^{-1} = \|A\| e^{i\theta}$$

$$\Rightarrow \quad \|A\|^2 = \left(1 + \frac{u^2 h^2}{\Delta^2} \sin^2 k\Delta \right)^{-1} < 1 \text{ always stable (damped)}$$

$$\theta = -\omega h = -ckh = -\tan^{-1}\left(\frac{uh}{\Delta} \sin k\Delta \right)$$

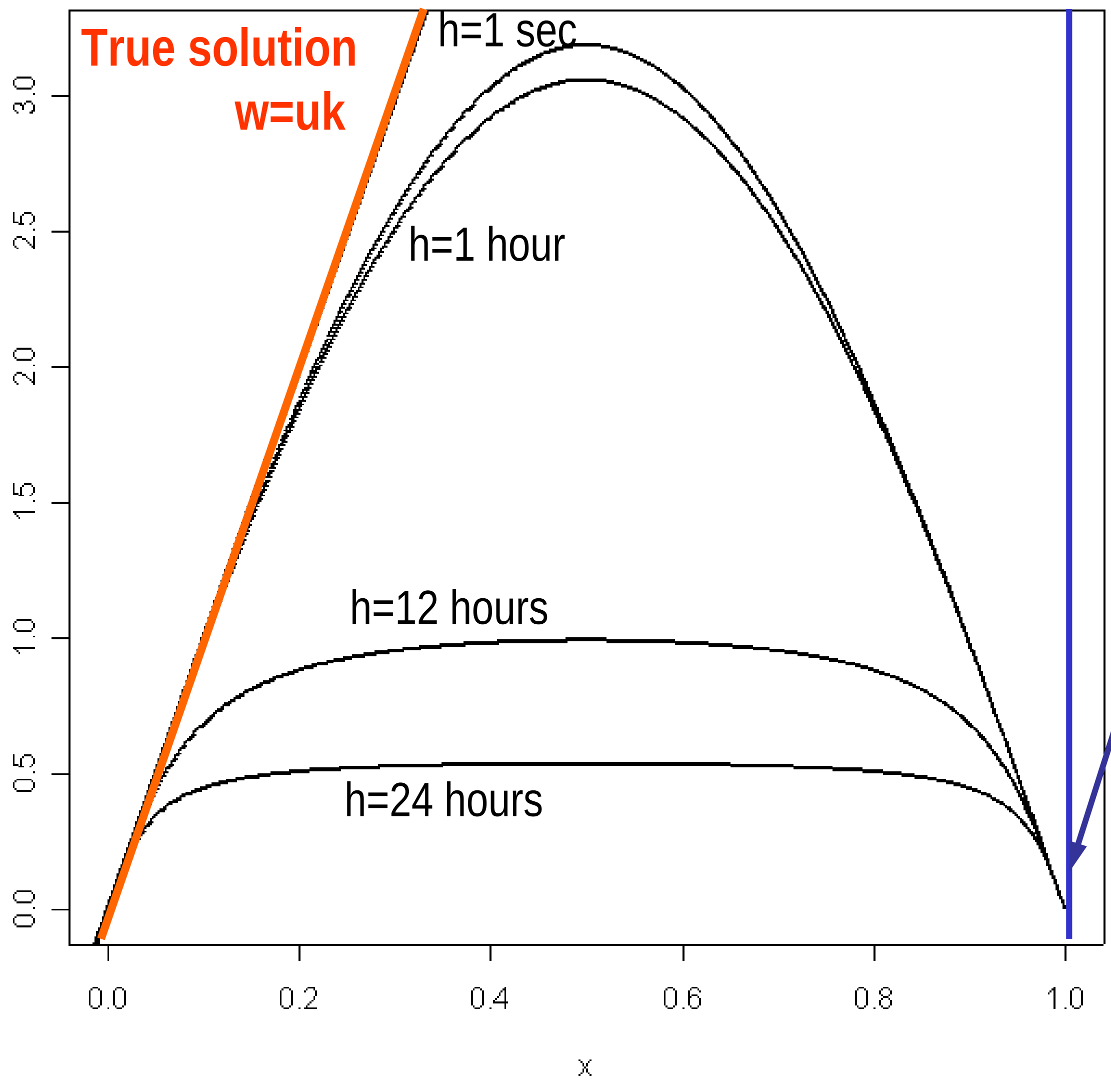
dispersion relation

Dependence of angular frequency on wavenumber

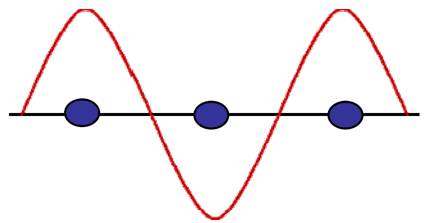
1.3 Slowing of propagation by implicit scheme

$$\frac{\omega\Delta}{\pi}$$

Dispersion relation for 1-d BTCS scheme u=10m/s d=100km



The 2-grid wave



has zero phase velocity!

$$\frac{k\Delta}{\pi} = \frac{2\Delta}{\lambda}$$

$$\frac{d\phi}{dt} = i\kappa\phi$$

Chapter 2 D.Durran "Numerical methods for wave equations in geophysical fluid dynamics"

Method	Order	Formula
→ Forward	1	$\phi^{n+1} = \phi^n + hF(\phi^n)$
→ Backward	1	$\phi^{n+1} = \phi^n + hF(\phi^{n+1})$
→ Asselin Leapfrog	1	$\phi^{n+1} = \overline{\phi^{n-1}} + 2hF(\phi^n)$ $\overline{\phi^n} = \phi^n + \gamma(\overline{\phi^{n-1}} - 2\phi^n + \phi^{n+1})$
→ Leapfrog	2	$\phi^{n+1} = \phi^{n-1} + 2hF(\phi^n)$
Adams–Bashforth	2	$\phi^{n+1} = \phi^n + \frac{h}{2}\left[3F(\phi^n) - F(\phi^{n-1})\right]$
→ Trapezoidal	2	$\phi^{n+1} = \phi^n + \frac{h}{2}\left[F(\phi^{n+1}) + F(\phi^n)\right]$
Runge–Kutta	2	$q_1 = hF(\phi^n)$, $\phi_1 = \phi^n + q_1$ $q_2 = hF(\phi_1) - q_1$, $\phi^{n+1} = \phi_1 + q_2/2$
Magazenkov	2	$\phi^n = \phi^{n-2} + 2hF(\phi^{n-1})$ $\phi^{n+1} = \phi^n + \frac{h}{2}\left[3F(\phi^n) - F(\phi^{n-1})\right]$
Leapfrog–Trapezoidal	2	$\phi_1 = \phi^{n-1} + 2hF(\phi^n)$ $\phi^{n+1} = \phi^n + \frac{h}{2}\left[F(\phi_1) + F(\phi^n)\right]$
Adams–Bashforth	3	$\phi^{n+1} = \phi^n + \frac{h}{12}\left[23F(\phi^n) - 16F(\phi^{n-1}) + 5F(\phi^{n-2})\right]$
Adams–Moulton	3	$\phi^{n+1} = \phi^n + \frac{h}{12}\left[5F(\phi^{n+1}) + 8F(\phi^n) - F(\phi^{n-1})\right]$
ABM Predictor–Corrector	3	$\phi_1 = \phi^n + \frac{h}{2}\left[3F(\phi^n) - F(\phi^{n-1})\right]$ $\phi^{n+1} = \phi^n + \frac{h}{12}\left[5F(\phi_1) + 8F(\phi^n) - F(\phi^{n-1})\right]$
Runge–Kutta	3	$q_1 = hF(\phi^n)$, $\phi_1 = \phi^n + q_1/3$ $q_2 = hF(\phi_1) - 5q_1/9$, $\phi_2 = \phi_1 + 15q_2/16$ $q_3 = hF(\phi_2) - 153q_2/128$, $\phi^{n+1} = \phi_2 + 8q_3/15$
Runge–Kutta	4	$q_1 = hF(\phi^n)$, $q_2 = hF(\phi^n + q_1/2)$ $q_3 = hF(\phi^n + q_2/2)$, $q_4 = hF(\phi^n + q_3)$ $\phi^{n+1} = \phi^n + (q_1 + 2q_2 + 2q_3 + q_4)/6$

TABLE 2.1. Summary of methods for the solution of ordinary differential equations. The second- and third-order Runge–Kutta methods are low-storage variants; $h = \Delta t$.

Method	Storage Factor (level)	Efficiency Factor	Amplification Factor ($\|A\|$)	Phase Error ($\frac{\theta}{\kappa\Delta t}$)	Max s
Forward	2	0	$1 + \frac{s^2}{2}$	$1 - \frac{s^2}{3}$	0
Backward	*	∞	$1 - \frac{s^2}{2}$	$1 - \frac{s^2}{3}$	∞
Asselin Leapfrog	3	< 1	$1 - \frac{\gamma s^2}{2(1-\gamma)}$	$1 + \frac{(1+2\gamma)s^2}{6(1-\gamma)}$	< 1
Leapfrog	2	1	1	$1 + \frac{s^2}{6}$	1
Adams–Bashforth–2	3	0	$1 + \frac{s^4}{4}$	$1 + \frac{5}{12}s^2$	0
Trapezoidal	*	∞	1	$1 - \frac{s^2}{12}$	∞
Runge–Kutta–2	2	0	$1 + \frac{s^4}{8}$	$1 + \frac{s^2}{6}$	0
Magazenkov	3	0.67	$1 - \frac{s^4}{4}$	$1 + \frac{s^2}{6}$	0.67
Leapfrog–Trapezoidal	3	0.71	$1 - \frac{s^4}{4}$	$1 - \frac{s^2}{12}$	1.41
Adams–Bashforth–3	4	0.72	$1 - \frac{3}{8}s^4$	$1 + \frac{289}{720}s^4$	0.72
Adams–Moulton–3	*	0	$1 + \frac{s^4}{24}$	$1 - \frac{11}{720}s^4$	0
ABM Predictor–Corrector–3	4	0.60	$1 - \frac{19}{144}s^4$	$1 + \frac{1243}{8640}s^4$	1.20
Runge–Kutta–3	2	0.58	$1 - \frac{s^4}{24}$	$1 + \frac{s^4}{30}$	1.73
Runge–Kutta–4	4[†]	0.70	$1 - \frac{s^6}{144}$	$1 - \frac{s^4}{120}$	2.82

[†] A storage factor of 3 may be achieved following the algorithm of Blum (1962).

small s assumed!

TABLE 2.2. Characteristics of the schemes listed in Table 2.1. The amplification factor and relative phase change are for well-resolved solutions to the oscillation equation, and $s = \kappa\Delta t$. "Max s" is the maximum value of $\kappa\Delta t$ for which the solution is nonamplifying. The storage and efficiency factors are defined in the text. No storage factor is given for implicit schemes.

$\phi^{n+1} = A\phi^n$ (Von Neumann stability analy

$A = |A| e^{i\theta}$

$\frac{d\phi}{dt} = i\kappa\phi$

$s = \kappa\Delta t$

Exercise: repeat some of these calculations but for $\frac{d\phi}{dt} = -\nu\phi$

1.3 Summary of main points

- Weather prediction and climate simulation is limited by the amount of computer time available
- Stability, accuracy and speed are the main concerns when developing numerical schemes
- CFL stability analysis shows that higher spatial resolution simulations require smaller time steps
- Methods have been developed to
  - filter out fast waves (e.g. sound waves, v.prop gravity waves)
  - slow down fast waves (e.g. semi-implicit schemes)
  - treat fast advection by moving with the flow (semi-lagrangian)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Practical assignment: programming style

- **Careful design** – think before you code!
- **Structured code** – modular programming. Break problem up into smaller tasks and then use subroutines to do the tasks. Keep information flow local.
- **Systematic debugging** – develop and use diagnostic tools. Use print and stop statements. Dry run code in your head. Pay particular attention to the tricky bits (e.g. boundaries).
- **Good documentation –** clear concise and informative use of comment statements, variable names, labels.