

MSc Module MTMM14:

Numerical modelling of atmospheres and oceans

Staggered schemes

3.1 Staggered time schemes (semi-implicit)

3.2 Staggered spatial grid discretisations

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- 1.3 Dynamical equations for the unforced fluid

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3.1 Staggered time schemes

Basic idea:

Represent time derivative by centered time finite difference (CT) and then put different terms of the equation on different time levels. (mixed scheme)

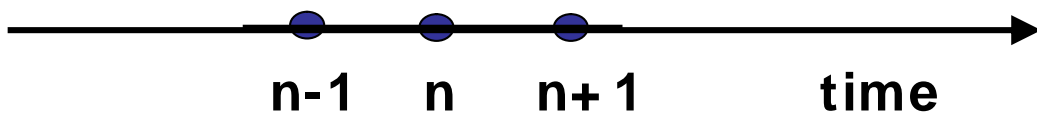
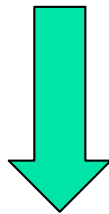
Why?

- Different equations have different stability properties (e.g. diffusion equation is unstable for CT and advection equation is unstable for FT scheme).
- Treat some of the terms implicitly in order to slow down simulated waves and get round CFL criterion (semi-implicit)

3.1 Example 1: Implicit diffusion schemes

Example:

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = \nu \nabla^2 \xi$$



$$\frac{\xi^{(n+1)} - \xi^{(n-1)}}{2h} + u^{(n)} \cdot \nabla \xi^{(n)} = \nu \nabla^2 \xi^{(n+1)}$$

- centred for advection (conditionally stable)
- backward for diffusion (implicit & fully stable)

3.1 Semi-implicit gravity wave schemes

Key concept:

slow down fast gravity waves by treating them implicitly while treating other terms explicitly. Split the tendency terms and put them on different time levels.

Example: shallow water equations

$$\frac{\partial \Psi}{\partial t} = -Q\Psi$$
$$Q = \begin{pmatrix} 0 & -f & \partial_x \\ f & 0 & \partial_x \\ \partial_x & \partial_x & 0 \end{pmatrix}$$
$$\Psi = (u, v, \phi)^T$$

3.1 Semi-implicit gravity wave schemes

Split the operator:

$$Q = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_x \\ \partial_x & \partial_x & 0 \end{pmatrix}$$
$$= R(\text{Rossby}) + G(\text{gravity})$$

- Centered scheme (explicit)

$$\Psi^{(n+1)} - \Psi^{(n-1)} = -2h(R + G)\Psi^{(n)}$$

- Backward scheme (fully implicit)

$$\Psi^{(n+1)} - \Psi^{(n-1)} = -2h(R + G)\Psi^{(n+1)}$$

- Mixed scheme (semi-implicit)

$$\Psi^{(n+1)} - \Psi^{(n-1)} = -2h(R\Psi^{(n)} + G\bar{\Psi}^{(n)})$$

3.1 Semi-implicit gravity wave schemes

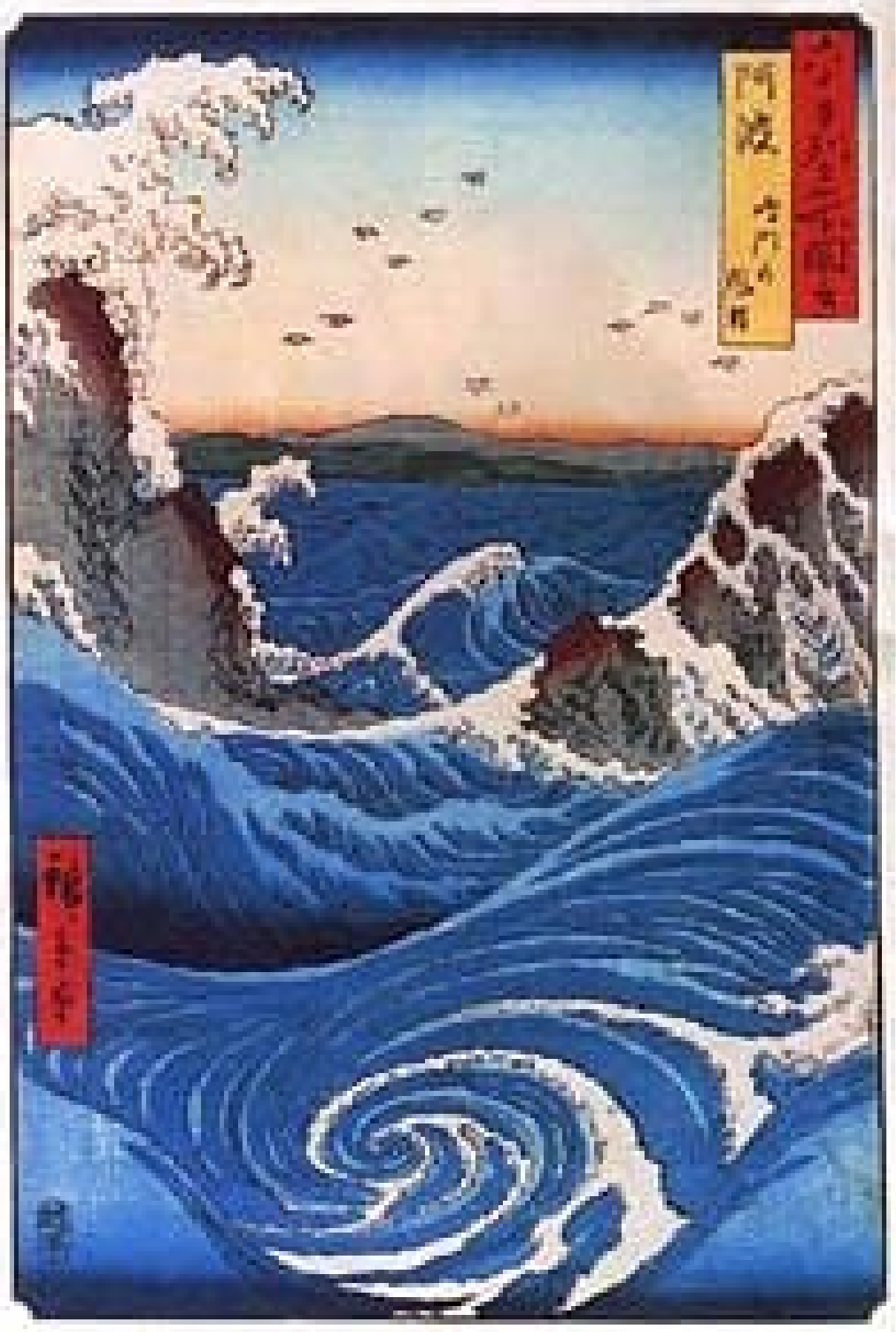
- The sparse operator $1+hG$ is a lot easier to invert than is $1+2h(R+G)$ to obtain a simple Helmholtz equation for height field. Rank of matrix to be inverted is 1/3 that of fully implicit scheme.
- Fast sparse matrix techniques can be used to invert $(1+hG)$
- Causes gravity waves to be treated implicitly and so larger time steps can be used without running into CFL problems

Robert, A., 1982: A semi-Lagrangian and semi-implicit numerical integration scheme for the primitive meteorological equations. J. Meteorol. Soc. Japan, 60, 319-325.

Robert, A., T.L. Yee, and H. Ritchie, 1985: A semi-Lagrangian and semi-implicit numerical integration scheme for multilevel atmospheric models. Mon. Weather Rev., 113, 388-394.

3.1 Summary of main points

- To obtain numerical stability, it can be advantageous to treat different tendency terms using different time schemes
- Example 1: implicit diffusion schemes
- Example 2: semi-implicit gravity wave schemes.



浪の巻物

大波

大波

3.2 Staggered spatial differencing schemes

Key idea:

Put different prognostic variables on different spatial grid points.

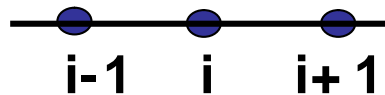
Why?

- More accurate (can halve the grid spacing!)
- Can improve the propagation of coupled waves so that they don't get stuck on the grid (e.g. gravity waves)
- Can reduce the amplitude of spurious gravity waves in initialisation (better balanced flow).

3.2 Example: 1-d gravity waves

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial u}{\partial x} = 0$$



x-space

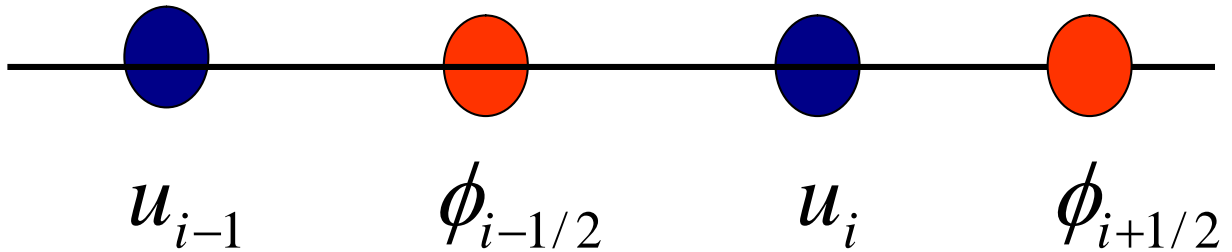
$$\frac{\partial u_i}{\partial t} + \frac{(\phi_{i+1} - \phi_{i-1}))}{2\Delta} = 0$$

$$\frac{\partial \phi_i}{\partial t} + \frac{(u_{i+1} - u_{i-1}))}{2\Delta} = 0$$

$$\Rightarrow \frac{\partial^2 \phi_i}{\partial t^2} = \frac{1}{4\Delta^2} (\phi_{i+2} + \phi_{i-2} - 2\phi_i)$$

- even and odd grid points are decoupled!
- grid spacing $2d$ not d (so less accurate)

3.2 Example: staggered 1-d grid



$$\frac{\partial u_i}{\partial t} + \frac{(\phi_{i+1/2} - \phi_{i-1/2})}{\Delta} = 0$$

$$\frac{\partial \phi_{i+1/2}}{\partial t} + \frac{(u_{i+1} - u_i)}{\Delta} = 0$$

$$\Rightarrow \frac{\partial^2 \phi_{i+1/2}}{\partial t^2} = \frac{1}{\Delta^2} (\phi_{i+3/2} + \phi_{i-1/2} - 2\phi_{i+1/2})$$

- grids are no longer decoupled
- grid spacing is d not $2d$ (so more accurate)

Key idea:

put mass fields (e.g. pressure) on one grid and motion fields (e.g. u velocity) on the other grid.

3.2 Higher dimensional staggered grids

- Staggered in space (Arakawa grids)
- Staggered in space and time (e.g. the Eliassen 1956 grid – Arakawa D grid staggered in time)
- Some short-hand operator notation:

$$(\delta_x \Psi)_{ij} = \Psi_{i+1/2,j} - \Psi_{i-1/2,j}$$

$$(\overline{\Psi}^x)_{ij} = \frac{1}{2} (\Psi_{i+1/2,j} + \Psi_{i-1/2,j})$$

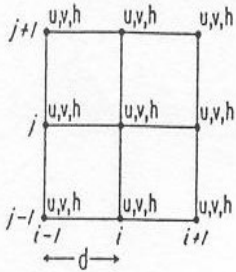
$$(\overline{\Psi}^{xy})_{ij} = \frac{1}{2} (\overline{\Psi}^y_{i+1/2,j} + \overline{\Psi}^y_{i-1/2,j})$$

More information:

http://www.ecmwf.int/newsevents/training/rcourse_notes/NUMERICAL_METHODS/index.html

Arakawa grids (Winnighoff 1968, Arakawa and Lamb 1977)

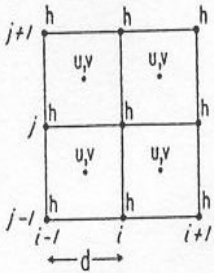
Scheme A:



$$\begin{aligned} \partial u / \partial t - f v + (g/d)(\overline{\delta_x h})^x &= 0, \\ \partial v / \partial t + f u + (g/d)(\overline{\delta_y h})^y &= 0, \\ \partial h / \partial t + (H/d)[(\overline{\delta_x u})^x + (\overline{\delta_y v})^y] &= 0; \end{aligned}$$

Even and odd grids decoupled.
Effective grid spacing $2d$ and
so not as accurate.

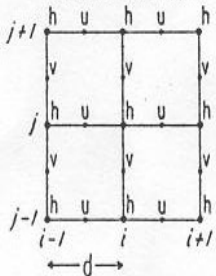
Scheme B:



Try to do this one yourself!

E.g. U.K. Met. office
(forward/backward scheme is easiest in this grid)

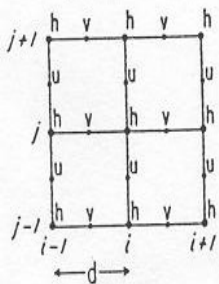
Scheme C:



$$\begin{aligned} \partial u / \partial t - f \bar{v}^{xy} + (g/d)(\delta_x h) &= 0, \\ \partial v / \partial t + f \bar{u}^{xy} + (g/d)(\delta_y h) &= 0, \\ \partial h / \partial t + (H/d)[(\delta_x u) + (\delta_y v)] &= 0; \end{aligned}$$

E.g. HIRLAM, DWD, oceanographers
(Semi-implicit is easiest on this grid)

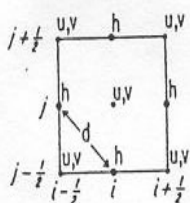
Scheme D:



$$\begin{aligned} \partial u / \partial t - f \bar{v}^{xy} + (g/d)(\overline{\delta_x h})^{xy} &= 0, \\ \partial v / \partial t + f \bar{u}^{xy} + (g/d)(\overline{\delta_y h})^{xy} &= 0; \\ \partial h / \partial t + (H/d)[(\overline{\delta_x u})^{xy} + (\overline{\delta_y v})^{xy}] &= 0; \end{aligned}$$

E.g. DNMI
NCEP "nested grid" model
(lots of averaging)
Staggered in time gives Eliassen grid.

Scheme E:



$$\begin{aligned} \partial u / \partial t - f v + (g/d^*)(\delta_x h) &= 0, \\ \partial v / \partial t + f u + (g/d^*)(\delta_y h) &= 0, \\ \partial h / \partial t + (H/d^*)(\delta_x u) + (\delta_y v) &= 0; \end{aligned}$$

E.g. NCEP "ETA" model
(Note: Egrid is like rotated Bgrid)
 45°

Arakawa and Lamb (1977)

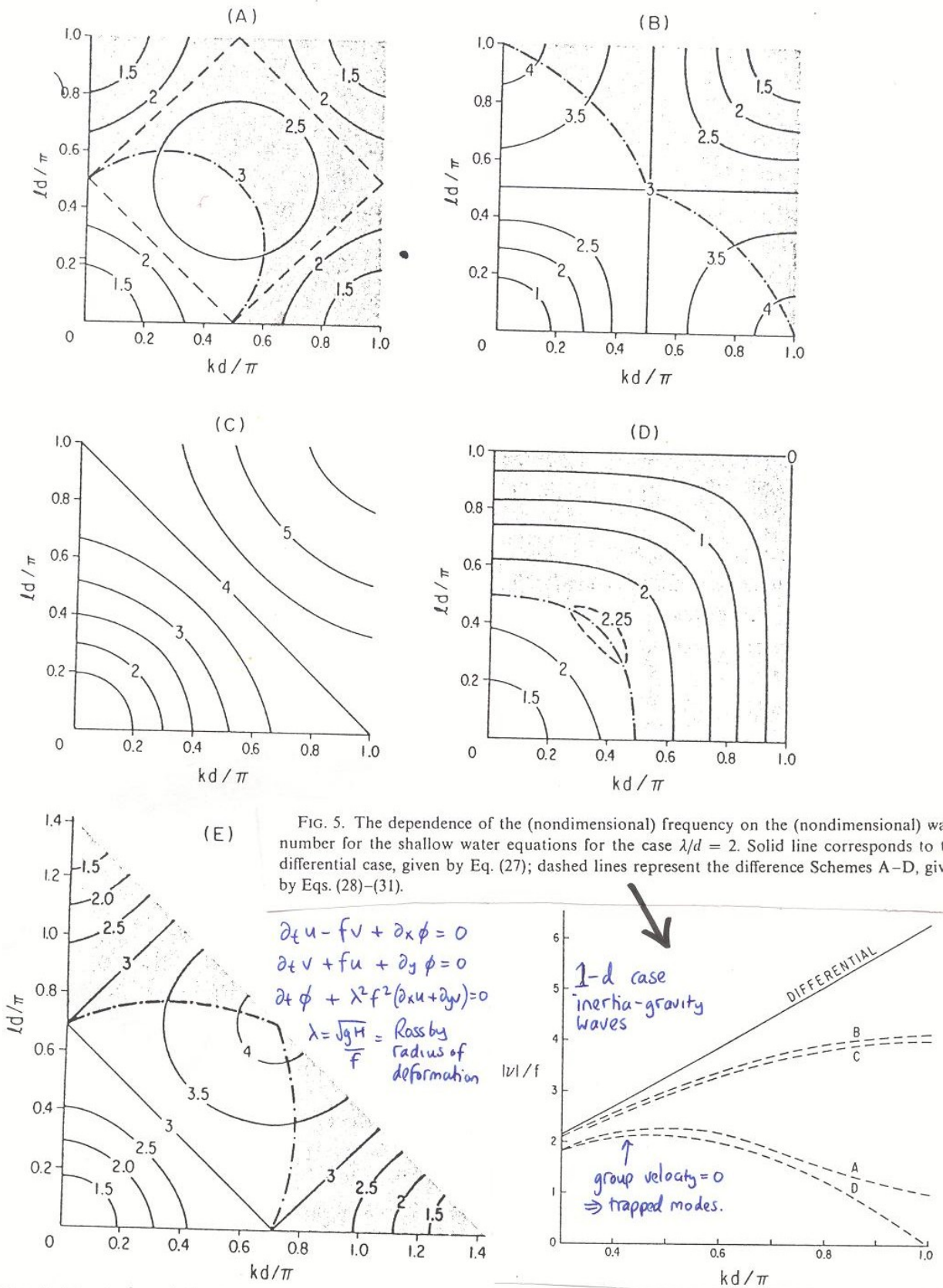


FIG. 8. Contours of the (nondimensional) frequency $|v|/f$ for Schemes A-E, as a function of the (nondimensional) horizontal wave numbers for the shallow water equations, for fixed $\lambda/d = 2$. Chain lines show the position of maximum values of the function for a range of the ratio l/k .

3.2 Vertical difference schemes

- Choice of “altitude” variable:
 - Height z
 - Pressure p
 - Sigma coordinate p/p_{surf}
 - Hybrid coordinate
- Choice of grid
 - Unstaggered – all variables on all levels
 - Lorenz grid
 - Charney-Phillips grid

Simmons, A. J. and Burridge, D. M. (1981).
An energy and angular-momentum conserving
vertical finite-difference scheme and hybride
vertical coordinates. *Mon. Weather. Rev.*,
109:758-766.

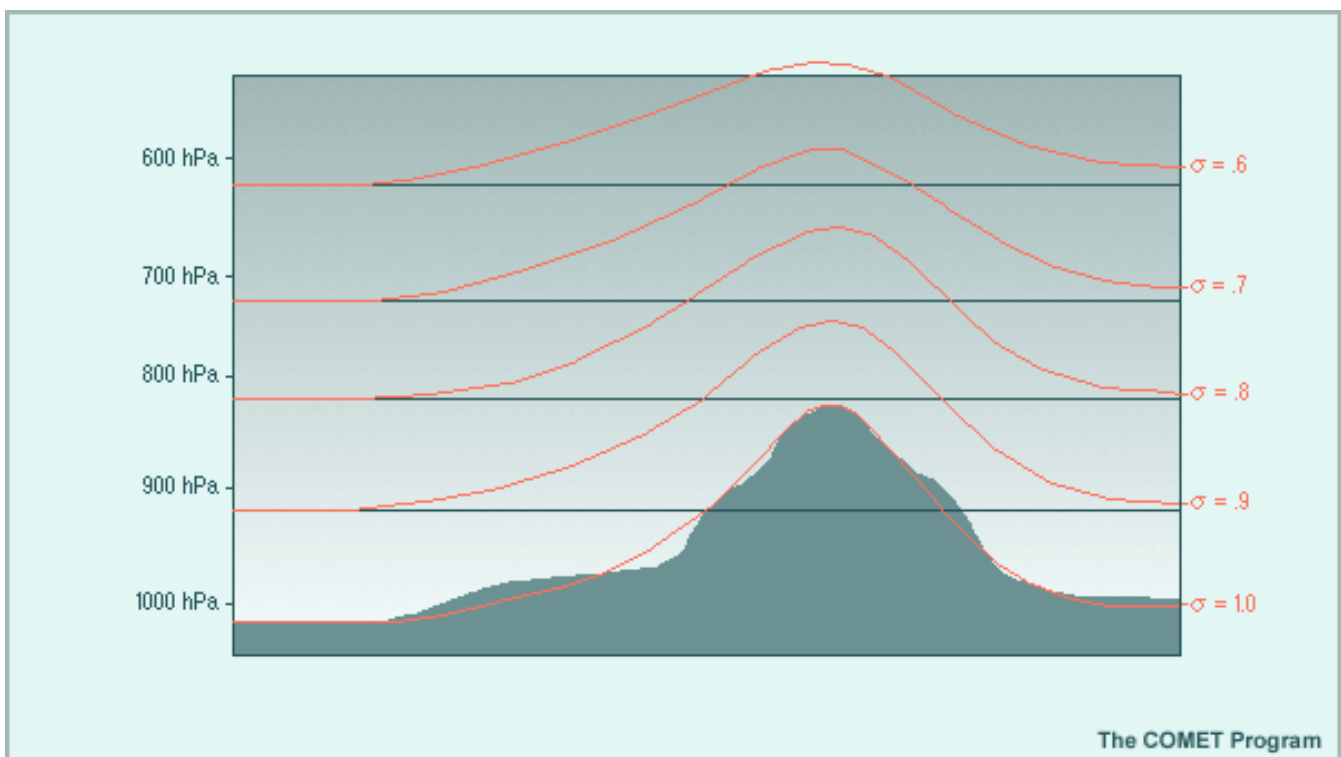
3.2 Choice of vertical coordinate

Height z – intuitive but isn't the simplest for the primitive equations.

Pressure p – good for primitive equations but can intersect the ground

Sigma coordinate $\sigma = p/p_s$ – terrain following but can produce numerical artefacts at high altitudes

Hybrid coordinate – compromise solution.



3.2 Hybrid vertical coordinates

Following Simmons and Burridge (1981) consider a hybrid vertical coordinate $z(p, p_s)$ that is a function of pressure p and surface pressure p_s .

Define the function in terms of $A(\cdot)$ and $B(\cdot)$:

$$p = A(z) + B(z)p_s$$

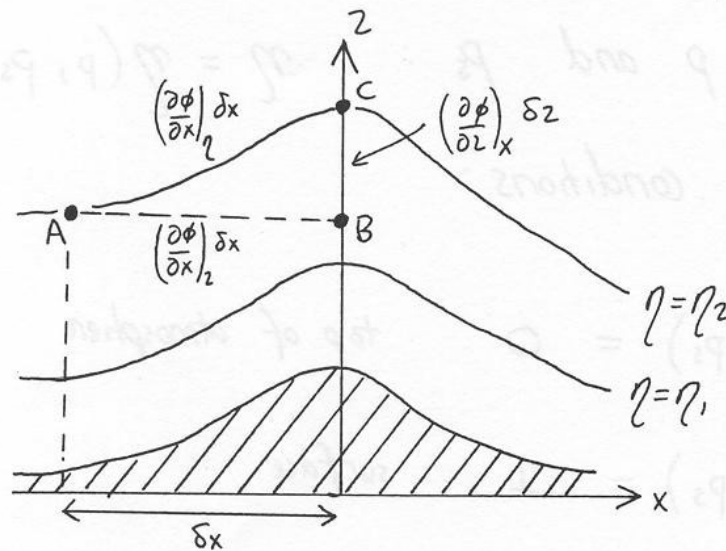
Examples:

- $A(z) = z$ $B(z) = 0 \rightarrow z = p$ pressure coordinate
- $A(z) = 0$ $B(z) = z \rightarrow z = p/p_s$ sigma coordinate

Assume z satisfies the following conditions:

- Zero at top of atmosphere: $z(0, p_s) = 0$
- One at bottom of atmosphere: $z(p_s, p_s) = 1$
- Monotonic function of pressure: $dz/dp > 0$

Choose a suitable vertical coordinate $\eta(x, y, z, t)$



Difference in field ϕ at points C and A

$$\Delta\phi = \phi_C - \phi_A = \left(\frac{\partial\phi}{\partial x} \right)_\eta \delta x$$

$$= \phi_B - \phi_A + \phi_C - \phi_B$$

$$= \left(\frac{\partial\phi}{\partial x} \right)_z \delta x + \left(\frac{\partial\phi}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_\eta \delta x$$

$$\left(\frac{\partial\phi}{\partial s} \right)_\eta = \left(\frac{\partial\phi}{\partial s} \right)_z + \left(\frac{\partial\phi}{\partial z} \right)_s \left(\frac{\partial z}{\partial s} \right)_\eta$$

$$\left(\frac{\partial\phi}{\partial \eta} \right)_s = \left(\frac{\partial\phi}{\partial z} \right)_s \left(\frac{\partial z}{\partial \eta} \right)_s \quad s=x, y, t$$

ϕ is any spatial field

↑ can use these equations to write adiabatic equations for any choice of vertical coordinate.

3.2 Staggered vertical grids

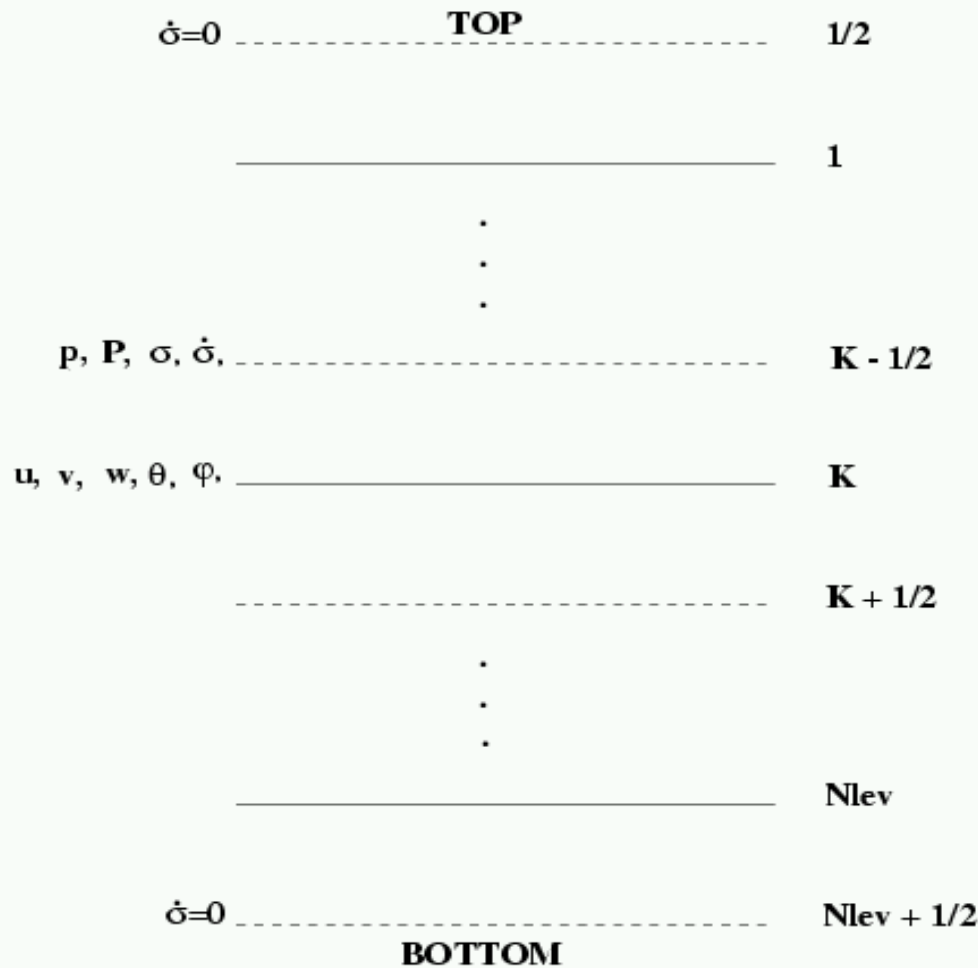


Figure 2: The equally-spaced sigma coordinate system used in the vertical discretization. The prognostic variables reside at the full levels (solid lines) and the diagnostic variables are at the half levels (dashed lines). No-flux boundary conditions are used at the top and bottom of the atmosphere.

Arakawa, A. and Moorthi, S. 1988:

Baroclinic Instability in Vertically Discrete Systems,

Journal of the Atmospheric Sciences: Vol. 45, No. 11,
pp. 1688–1708.

3.2 Summary of main points

- Schemes can be improved by careful choice of grid
- Can use imaginative coordinates – stretched grids in lat-lon and vertical
- Can stagger different variables on different grids to remove 2-grid waves and improve accuracy.
- All these are “structured meshes” but can also consider unstructured adaptive meshes that adapt to the flow ...