Exercise 3. Probability concepts

1. Let X be the random variable representing the outcome of a single roll of a fair, six-sided die. Write down the sample space for X. Draw a Venn diagram showing the following three events: A_1, an odd number; A_2, an even number; and A_3, a number greater than four.

   (a) What are the values of the probabilities $\Pr(A_1)$, $\Pr(A_2)$ and $\Pr(A_3)$?

   $\Pr(A_1) =$
   $\Pr(A_2) =$
   $\Pr(A_3) =$

   (b) Which pairs of the events do you think are mutually independent? Which pairs do you think are mutually exclusive? Find $\Pr(A_1 \text{ and } A_2)$, $\Pr(A_1 \text{ and } A_3)$ and $\Pr(A_2 \text{ and } A_3)$, and check which pairs are independent and which are exclusive.

   $\Pr(A_1 \text{ and } A_2) =$
   $\Pr(A_1 \text{ and } A_3) =$
   $\Pr(A_2 \text{ and } A_3) =$

(c) Find the conditional probabilities $\Pr(A_1 \mid A_3)$ and $\Pr(A_2 \mid A_3)$. Are these what you expected? Compute the conditional probabilities $\Pr(A_3 \mid A_1)$ and $\Pr(A_3 \mid A_2)$ using Bayes's Theorem.

$\Pr(A_1 \mid A_3) =$
$\Pr(A_2 \mid A_3) =$
$\Pr(A_3 \mid A_1) =$
$\Pr(A_3 \mid A_2) =$

2. To get a white Christmas in London there needs to be precipitation and the boundary layer needs to be below freezing point. If the probability of precipitation on Christmas day is $1/2$ and the probability that the boundary layer will be below freezing is $1/3$ then calculate the probability of a white Christmas assuming independence of these two events. If observations show that a white Christmas in London happens, on average, once every 10 years then calculate the conditional probability of precipitation given that the boundary layer is below freezing and compare it with the unconditional probability of precipitation.

3. Generate a random sample of size 10 for the random variable X in question 1. Write down the proportion of times event A_3 occurs. Repeat for samples of sizes 100, 1000, 10 000 and 100 000. Are the results consistent with the Law of Large Numbers?

10	100	1000	10 000	100 000

4. Using your sample of size 100, compute the number of times each of the possible values of X occurs. Compare these with the expected number for each value. Think of some other ways to check that `R` is able to generate seemingly random numbers, and try them out on your sample.

	1	2	3	4	5	6
Observed						
Expected						

5. What are the values of the expectation and variance of X?

$E(X) =$
$\mathrm{Var}(X) =$

Let Y be the random variable representing the outcome of a second, *independent* roll of the same die. Write down the values of the following quantities without doing any calculations.

$E(X+Y) =$	$E(XY) =$
$\mathrm{Cov}(X,Y) =$	$\mathrm{Cor}(X,Y) =$
$\mathrm{Var}(X+Y) =$	$\mathrm{Var}(X-Y) =$

6. What would you guess is the chance of finding at least two people with the same birthday (not necessarily year) out of a sample of 30 people? By counting the number of possible birthdays for each person, calculate the probability that all of the 30 people have different birthdays. What is the probability that two or more of the people share a birthday? Think of reasons for any difference with your guess. An interactive demonstration of this 'Birthday Problem' is at `www-stat.stanford.edu/~susan/surprise` and an explanation is at `www.mste.uiuc.edu/reese/birthday`.