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THE INFLUENCE OF THE EARTH'S ROTATION UPON THE RELATIVE MOTION OF BODIES NEAR ITS SURFACE.

BY W. FERREL.

If a body upon or near the earth's surface receive a motion relatively to the earth, either by means of a single impulse or by a continually acting force, this motion, combined with the rotatory motion of the earth, gives rise to a deflecting force relatively to the earth, which causes a different relative motion from that of a body acted upon in a similar manner upon the earth at rest. It is proposed, in this paper, to examine a few of the effects produced by this deflecting force.

Let x, y and z be three rectangular coördinates having their origin at the center of the earth, x corresponding with the axis; and $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ be the forces which act respectively in the directions of these ordinates. We shall then have

$$\frac{ddx}{dt^2} = \boldsymbol{P}\,; \quad \frac{ddy}{dt^2} = \boldsymbol{Q}\,; \quad \frac{ddz}{dt^2} = \boldsymbol{R}. \qquad [1]$$

Let r = the distance of the body from the earth's center
θ = its polar distance
ϖ = its longitude [printed π in the fractions]
nt = the rotatory motion of the earth.

We shall then have

$$x = r\cos\theta\,; \; y = r\sin\theta\cos(nt+\varpi)\,; \; z = r\sin\theta\sin(nt+\varpi) \qquad [2]$$

Substituting the second differential of the value of x in the first of equations [1], we get

$$\cos\theta\frac{ddr}{dt^2} - 2\sin\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} - r\cos\theta\frac{d\theta^2}{dt^2} - r\sin\theta\frac{dd\theta}{dt^2} = \boldsymbol{P} \qquad [3]$$

Substituting in like manner the second differentials of y and z in the last two of equations [2], and multiplying the former by $\cos(nt+\varpi)$ and the latter by $\sin(nt+\varpi)$, and adding, we get

$$\sin\theta\frac{ddr}{dt^2} + 2\cos\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} - r\sin\theta\frac{d\theta^2}{dt^2} + r\cos\theta\frac{dd\theta}{dt^2} - r\sin\theta\left(n+\frac{d\pi}{dt}\right)^2 = \cos(nt+\varpi)\,\boldsymbol{Q} + \sin(nt+\varpi)\,\boldsymbol{R} \qquad [4]$$

Multiplying equation [3] by $\cos\theta$ and equation [4] by $\sin\theta$, and adding, we get the first of equations [5]. Multiplying the former by $\sin\theta$ and the latter by $\cos\theta$, and subtracting, we get the second of equations [5]. Again, after substituting the second differentials of y and z, as stated above, in the last two of equations [1], if we multiply the former by $\sin(nt+\varpi)$ and the latter by $\cos(nt+\varpi)$, and subtract, we get the last one of the following equations.

$$\frac{ddr}{dt^2} - r\frac{d\theta^2}{dt^2} - r\sin^2\theta\left(n+\frac{d\pi}{dt}\right)^2 = \cos\theta\,\boldsymbol{P} + \sin\theta\cos(nt+\varpi)\,\boldsymbol{Q} + \sin\theta\sin(nt+\varpi)\,\boldsymbol{R}\,;$$

$$-r\frac{dd\theta}{dt^2} - 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\sin\theta\cos\theta\left(n+\frac{d\pi}{dt}\right)^2 = \sin\theta\,\boldsymbol{P} - \cos\theta\cos(nt+\varpi)\,\boldsymbol{Q} - \cos\theta\sin(nt+\varpi)\,\boldsymbol{R}\,; \qquad [5]$$

$$-r\sin\theta\frac{dd\pi}{dt^2} - 2\sin\theta\left(n+\frac{d\pi}{dt}\right)\frac{dr}{dt} - 2r\cos\theta\left(n+\frac{d\pi}{dt}\right)\frac{d\theta}{dt} = (\sin nt+\varpi)\,\boldsymbol{Q} - \cos(nt+\varpi)\,\boldsymbol{R}.$$

The preceding equations may be applied to the relative motions of a body acted upon by any forces whatever. We shall first apply them to the motions of projectiles; and in the application, we shall neglect all quantities of the order of the earth's ellipticity; and, also, as the range of motion is generally small, all quantities of the order of the range of motion compared with the earth's radius. We shall also, in integrating, consider $\sin\theta$ and $\cos\theta$ constant quantities. We shall then have

$$\boldsymbol{P} = -\cos\theta\, gr^{-2},$$
$$\boldsymbol{Q} = -\sin\theta\cos(nt+\varpi)\, gr^{-2},$$

and

$$\boldsymbol{R} = -\sin\theta\sin(nt+\varpi)\, gr^{-2},$$

in which g = the force of gravity at the distance of unity. Substituting these values in the righthand members of equations [5], the first becomes $-gr^{-2}$, and the other two become 0. As $\frac{d\pi}{dt}$ and $\frac{d\theta}{dt}$ are generally small in comparison with n, and very nearly constant, we shall, in integrating, use the initial values of those quantities. We may hence deduce from equations [5],

$$\alpha = u\,t - \tfrac{1}{2} g r'^{-2} t^2 + \frac{v^2}{2r'} t^2 + \frac{\sin\theta\,(n'+w)^2}{2r'} t^2$$

$$\beta = v\,t - \frac{uv}{r'} t^2 + \frac{g' v}{3 r'} t^3 + \frac{\cos\theta\,(n'+w)^2}{2r'} t^2 \qquad [6]$$

$$\gamma = w\,t - \frac{u}{r'}(n'+w) t^2 + \frac{n'+w}{3r'} g' t^3 - \frac{\cos\theta\,(n'+w) v}{r' \sin\theta} t^2\,;$$

in which

$\alpha = \int dr =$ the vertical motion in the time t
$\beta = \int r' d\theta =$ the linear motion in latitude
$\gamma = \int r' \sin\theta\, d\varpi =$ the linear motion in longitude
$u =$ the initial vertical velocity
$v =$ the initial velocity in latitude
$w =$ the initial velocity in longitude
$r' =$ the value of r at the earth's surface
$n' = r' \sin\theta\, n =$ the linear velocity of the earth's surface
$g' = g\, r'^{-2} - r' \sin^2\theta\, n^2 =$ gravity at the earth's surface, diminished by the effect of the earth's rotation.

If in these equations we put $n' = 0$, they will be the equations of motion upon the earth without a rotatory motion. If we make r' infinite, we have the ordinary equations of parabolic motion, in which the directions of gravity are considered parallel. If we retain $\frac{\sin\theta\, n'^2}{2r'} t^2 = \frac{1}{2} \sin^2\theta\, n$, contained in the last term of the first equation, which depends upon the earth's rotation alone, we still have the equations of motion in a parabola, but in this case the effect of gravity is diminished by that term; and retaining also $\frac{\cos\theta\, n^2}{2r'} t^2$, contained in the last term of the second equation, the direction of the axis of the parabola is changed by an angle ψ, in which $\tan\psi = \frac{\sin\theta \cos\theta}{g r^{-1} - \sin^2\theta n^2}$. The effect of n' in the preceding equations, in all ordinary cases of projectiles, is very small. If a projectile were thrown 2 miles, in any direction, in 10 seconds, the effect of the earth's rotation would deflect it from that direction, in the latitude of Cambridge, about 8 feet to the right. From the first of the preceding equations it may be seen that if w is plus, that is, if the initial velocity is east, the projectile will ascend a little higher than when it is minus, or the initial velocity is west, all other circumstances being the same.

If in equations [6] we put v and $w = 0$, they become the equations of an ascending or descending body; and putting also $u = 0$, they become the equations of a falling body without any initial velocity. In the latter case they become

$$\alpha = -\tfrac{1}{2}(g r'^{-2} - r' \sin^2\theta\, n^2)\, t^2$$
$$\beta = \tfrac{1}{2} r' \sin\theta \cos\theta\, n^2 t^2 \qquad [7]$$
$$\gamma = \tfrac{1}{3} \sin\theta\, n\, g' t^3$$

The second of the preceding equations shows that a body falls south of the direction of the earth's attraction. As the first two equations evidently give the equation of a plumb-line, a falling body cannot deviate to the north or south of it. If, however, the falling body is resisted by the atmosphere, so that α is not in proportion with t^2, they no longer give the equation of a straight line, and the effect evidently is to throw the body a little to the south of a plumb-line in the northern hemisphere, but north of it in the southern. Hence the observed deviation of falling bodies in the northern hemisphere to the south of a plumb-line, is caused by the resistance of the atmosphere together with the effect of the earth's rotation. The last of equations [7] shows that a falling body deviates to the east of a plumb-line, and that it falls in a curve convex towards it. The value of $n = \frac{2\pi}{(23\times60+56)\times60} = .00007292$. With this value of n, this equation gives about 7 inches for the eastern deviation of a body, in the latitude of Cambridge, in falling 10 seconds.

If a body be free to move upon the surface of the earth regarded as a perfect sphere, its motions may be determined by the last two of equations [5], the terms containing $\frac{dr}{dt}$ vanishing. The term $r \sin\theta \cos\theta\, n^2$, contained in the first one of those equations, gives rise to the elliptical figure of the earth. Since the resultant of this force and the earth's attraction is perpendicular to the elliptical surface, these forces cannot affect the motions of a body upon this surface. Hence we have for the motions of a body upon the elliptical surface of the earth, using r for the radius of the earth regarded as constant,

$$-r\frac{dd\theta}{dt^2} + r\sin\theta\cos\theta\left(2n + \frac{d\pi}{dt}\right)\frac{d\pi}{dt} = \sin\theta\, \boldsymbol{P} - \cos\theta\cos(nt+\varpi)\, \boldsymbol{Q} - \cos\theta\sin(nt+\varpi)\, \boldsymbol{R}$$
$$-r\sin\theta\frac{dd\pi}{dt^2} - 2r\cos\theta\left(n + \frac{d\pi}{dt}\right)\frac{d\theta}{dt} = \sin(nt+\varpi)\, \boldsymbol{Q} - \cos(nt+\varpi)\, \boldsymbol{R}. \qquad [8]$$

If the body be acted upon by a central force $\boldsymbol{F}$ at the surface, and ϱ be the distance in arc of the body from the center of force, and μ the angle which ϱ makes with the meridian reckoned from the south towards the east, if we put $n = 0$, we can change θ to ϱ and ϖ to μ in the preceding equations, and thus deduce the equations of motion upon the spherical surface of the earth without rotation. Since gravity can produce no effect, the right member of the first equation becomes $\boldsymbol{F}$, and of the second equation 0. Hence they become

$$r\frac{dd\varrho}{dt^2} = r\sin\varrho\cos\varrho\frac{d\mu^2}{dt^2} - F$$
$$r\sin\varrho\frac{dd\mu}{dt^2} = -2r\cos\varrho\frac{d\varrho}{dt}\cdot\frac{d\mu}{dt} \qquad [9]$$

If to the right-hand members of these equations we add the forces arising from the terms in the preceding equations containing n, resolved in the direction of ϱ and perpendicular to it, and expressed in terms of ϱ and μ instead of θ and ϖ, we shall have the equations of the motion of a body upon the elliptical surface of the earth with a rotatory motion, disregarding effects of the order of the earth's ellipticity. The eqnations will then become

$$r\frac{dd\varrho}{dt^2} = r\sin\varrho\cos\varrho\frac{d\mu^2}{dt^2} + 2r\sin\varrho\cos\theta\, n\frac{du}{dt} - F$$
$$r\sin\varrho\frac{dd\mu}{dt^2} = -2r\cos\varrho\frac{d\varrho}{dt}\cdot\frac{d\mu}{dt} - 2r\cos\theta\, n\frac{d\varrho}{dt} \qquad [10]$$

If in these equations we put $F = 0$, they become the equations of the motion of a body when it is not acted upon by any superficial or tangential forces. If in these equations we then put $\frac{d\varrho}{dt} = 0$ and $\frac{dd\varrho}{dt^2} = 0$, the last equation, regarding $\cos\theta$ constant, gives $\frac{d\mu}{dt} = m$, a constant, which is the initial angular velocity. This value, substituted in the first equation, satisfies it when $\cos\varrho\, m = -2\cos\theta\, n$. [11]

When ϱ is small, $\cos\varrho$ may be put $= 1$. Hence when the range of motion is so small in comparison with the radius of the earth, that $\cos\theta$ may be regarded as a constant, these equations may be satisfied with a motion in the circumference of a circle of which the angular velocity m is uniform, and equal to twice the velocity of the earth's rotation multiplied by the sine of the latitude. Since the value of m is negative in the northern hemisphere and positive in the southern, this motion is from left to right in the former, and the contrary in the latter.

Since ϱ, when it is so small that $\cos\varrho$ may be put $= 1$, does not enter into the preceding equation, it depends for its value upon the initial linear velocity. Multiplying both members of the preceding equation by $r\sin\varrho$, and putting $r\sin\varrho = \varrho'$ = the linear distance, $\cos\varrho = 1$, and........ $r\sin\varrho\, m = v$ = the initial linear velocity, we obtain

$$v = -2\varrho'\cos\theta\, n. \qquad [12]$$

Hence *if a body receives a motion in any direction, it describes the circumference of a circle, if the range of motion is small, the radius of which is determined by the preceding equation; and the time of its performing a revolution is equal to the time of the earth's rotation divided by twice the sine of the latitude.*

The time of revolution, in the latitude of Cambridge, is $16^h\, 15^m$; and if $v = 100$ feet per second, the radius is 176.3 miles.

From what precedes, it is evident that if a body is moving in any direction, there is a force, arising from the earth's rotation, which always deflects it to the right in the northern hemisphere, and to the left in the southern. Since this deflecting force must always be at right angles with the direction, it cannot change v the linear velocity; and hence when the range of motion is great, the radius of curvature is always inversely as the sine of the latitude. The larger the range of motion, the more it deviates from a circle; but the curve must always be symmetrical on each side of the central meridian, and the body return to the point from which it started.

Equations [10] may be applied to the motions of a pendulum in which the oscillations are small in comparison with its length. Putting ϱ' = the distance of the pendulum from its vertical position of rest, we may put $r\sin\varrho = \varrho'$, $\cos\varrho = 1$, and $F = \frac{\varrho' g}{l}$, l being the length of the pendulum. These equations then become

$$\frac{dd\varrho'}{dt^2} = \varrho'\frac{d\mu^2}{dt^2} + 2\varrho'\cos\theta\, n\frac{d\mu}{dt} - \frac{\varrho' g}{l}$$
$$\varrho\frac{dd\mu}{dt^2} = -2\frac{d\varrho'}{dt}\cdot\frac{d\mu}{dt} - 2\cos\theta\, n\frac{d\varrho'}{dt} \qquad [13]$$

Putting $n = 0$, these equations become the differential equations of elliptic motion in which the body is attracted by a central force which varies as the distance. Hence the general motions of the pendulum, if the earth had no rotation, would be an ellipse of which the center would be the vertical position of the pendulum.

The preceding equations may be satisfied by the preceding motion in an ellipse which gyrates with an angular velocity m. The equations of such a motion are

$$\frac{dd\varrho'}{dt^2} = \varrho'\left(\frac{d\mu}{dt} - m\right)^2 - \frac{\varrho' g}{l}$$
$$\varrho'\frac{dd\mu}{dt^2} = -2\frac{d\varrho'}{dt}\left(\frac{d\mu}{dt} - m\right) \qquad [14]$$

If in these equations we put $m = -\cos\theta\, n$, they become the same as equations [13], omitting the very small term $\varrho' m^2$ in the first equation, which is of the order of the time of an oscillation compared with the time of the earth's rotation. Hence *the general motion of a pendulum is in an ellipse, which, in the northern hemisphere, gyrates from left to right in the time of the earth's rotation divided by the sine of the latitude.* Since the value of m is positive in the southern hemisphere, the gyration there is the contrary way.

The figure of this ellipse depends upon the initial value of $\frac{d\mu}{dt}$ and the initial direction of motion. If the initial direction is at right angles to ϱ', and such that $\left(\frac{d\mu}{dt} - m\right)^2 - \frac{g}{l} = 0$, the pendulum describes the circumference of a circle. If the initial value of $\frac{d\mu}{dt} = m$, equations [14] are reduced to

$$\frac{dd\varrho'}{dt^2} = -\frac{\varrho' g}{l}; \qquad \frac{dd\mu}{dt^2} = 0. \qquad [15]$$

The former is the equation of a simple pendulum vibrating in a right line; and the latter gives $\frac{d\mu}{dt}$ = a constant, which is its initial value m or $-\cos\theta\, n$. Hence, if the initial value of $\frac{d\mu}{dt} = -\cos\theta\, n$, the pendulum vibrates in a right line which performs a gyration in the same time and in the same manner as the ellipse in the more general case. If the pendulum receives no initial gyratory motion, its vibrations deviate a little from a right line.

We have seen that a body cannot move in any direction without being deflected to the right in the northern hemisphere, and to the left in the southern. It cannot, therefore, be attracted or forced towards a center, without at the same time receiving a gyratory motion around it. If, therefore, the lower strata of atmosphere flow in from all sides towards a center, on account of rarefaction produced by heat or any other cause, it must at the same time receive a gyratory motion around that center; and this motion must be from right to left in the northern hemisphere, and the contrary in the southern. This completely accounts for the well established fact that the motion of storms is gyratory; and that these gyrations, in the northern hemisphere, are always from right to left, and in the southern, from left to right.

The minimum amount of gyratory motion for a given amount of motion towards the center may be deduced from the last of equations [10]. Omitting the first term of the right member, which is at first small, and putting $r \sin\varrho \frac{d\mu}{dt} = v =$ the linear gyratory velocity, $r \sin\varrho = \varrho' =$ the linear distance, considering $\sin\varrho = \varrho$ and $\cos\varrho = 1$, we obtain

$$v = -2\cos\theta\, n\, \varrho' (\log\varrho' - \log\varrho''), \qquad [16]$$

in which ϱ'' is the value of ϱ' at the commencement of motion.

If we suppose the atmosphere, at the distance of 200 miles on all sides from the center, to flow 50 miles towards it, the preceding equation gives about 15 miles per hour for the amount of gyratory motion it must receive in that time, in the latitude of Cambridge, from the influence of the earth's rotation. A gyratory motion being once produced, it is very rapidly accelerated, especially near the center, by the effect of the term which we have omitted, which then acquires a considerable value. This term is independent of the earth's rotation, and depends entirely upon the action of the central force.

We hope to be able to give a complete application of these principles to the theory of storms at some future time.

Cambridge, 1857 *November* 10.

ELEMENTS AND EPHEMERIS OF THE SIXTH COMET OF 1857.

BY JAMES C. WATSON.

THE unfavorable state of the weather during the last two weeks having entirely prevented later observations of the new comet, discovered by VAN ARSDALE Nov. 10, I have computed, from observations made at Washington, Nov. 12, 15 and 17 (for which I am indebted to the kindness of Mr. FERGUSON), the following elements:

T = 1857 Nov. 16.985192 Washington M. T.
π = 45° 53′ 17″.7 } Mean Equinox 1857.0
☊ = 135 0 36.7 }
i = 37 44 18.8
$\log q$ = 0.004004

Motion retrograde.

The middle place is represented in the following manner:

Comp. — Obs.

$$\Delta\alpha\cos\delta = -8''.2, \quad \Delta\delta = -49''.7$$

These elements give the following

Ephemeris for Washington Mean Midnight.

1857	☋ α	☋ δ	log △
Dec. 2	20ʰ 1ᵐ 12ˢ	+17° 21′.9	9.9988
3	4 54	16 9.7	
4	8 22	15 1.3	
5	11 37	13 56.6	
6	14 41	12 55.4	0.0477
7	17 34	11 57.4	
8	20 18	11 2.2	
9	22 53	10 9.9	
10	25 20	9 20.3	0.0928
11	27 40	8 33.1	
12	29 53	7 48.2	
13	32 0	7 5.4	
14	34 2	6 24.6	0.1341
15	35 59	5 45.7	
16	37 51	5 8.6	
17	39 39	4 33.1	
18	41 23	3 59.2	0.1712
19	43 4	3 26.8	
20	44 42	2 55.7	
21	20 46 17	+ 2 25.9	

Ann-Arbor, 1857 *Dec.* 3. JAMES C. WATSON.