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On the Changes in the Spectral Distribution of Kinetic Energy for Twodimensional, Nondivergent Flow

By RAGNAR FJØRTOFT, University of Copenhagen

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Abstract

Total kinetic energy as well as total vorticity squared are integral quantities which cannot change in the course of time in a *twodimensional* flow of a homogeneous, nondivergent, and inviscid fluid when the fluid is isolated from the surroundings. The case is considered where the fluid is defined over the total region of the surface of a sphere. The nature of the changes in time of the spectral distribution of kinetic energy is discussed on the basis of the two conservation requirements mentioned above. It is found that only fractions of the initial energy can flow into smaller scales and that a greater fraction simultaneously has to flow to components with larger scales. The upper limits to the flow of kinetic energy into components with scales less than a given one are found. The conservation theorems are also used to discuss the stability of a certain stationary flow for a twodimensional motion which is not necessarily spherical. It is shown how important it is for the proof of stability that not only the kinetic energy of the disturbance is supposed to be small but also its vorticities.

In chapter II molecular viscosity is taken into account for the spherical flow. Finally some conclusive remarks are offered regarding the fundamental difference between two- and threedimensional flow.

1. Twodimensional spherical flow. Inviscid fluid

A twodimensional nondivergent flow of a homogeneous fluid defined over the total surface of a sphere is considered. Viscosity is neglected in the first place. The absolute motion of the fluid is then governed by the equations

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla_s \gamma - (\mathbf{v} \cdot \nabla_s \mathbf{v})_s \tag{1}$$

$$\mathbf{v} = -\nabla_s \psi \times \mathbf{k}. \tag{2}$$

Here

$\mathbf{v}$ = velocity
∇_s = spherical deloperator
γ = pressure over density
$-(\mathbf{v} \cdot \nabla_s \mathbf{v})_s$ = convective acceleration along the surface of the sphere
$\mathbf{k}$ = unit vectors perpendicular to the sphere

and

ψ = a univalued and twice differentiable function of the space coordinates. (3)

Total kinetic energy is obviously conserved for our fluid. Hence, with F denoting the total area of the surface of the sphere,

$$\int_F (\nabla_s \psi)^2 \, dF = \text{const.} \tag{4}$$

Eliminating $\nabla_s \gamma$ from (1) and using (2) one obtains

$$\frac{\partial \nabla_s^2 \psi}{\partial t} = -\mathbf{v} \cdot \nabla_s \nabla_s^2 \psi \tag{5}$$

where $\nabla_s^2 \psi$ represents the component of vorticity perpendicular to the surface of the sphere. Multiplying (5) with $\frac{1}{2} \nabla_s^2 \psi$ and integrating over F, one also obtains

$$\int_F (\nabla_s^2 \psi)^2 \, dF = \text{const.} \tag{6}$$

(4) and (6) express two conservation theorems. In the following it will be shown that it is possible to draw considerable information from them as to the character of the solution of (1), (2) when the conditions are known initially.

As is well known ψ may be written as a generally infinite sum of functions

$$\psi = \sum_{q=1}^{\infty} \psi_q \tag{7}$$

where ψ_q satisfies

$$\nabla_s^2 \psi_q + a_q \psi_q = 0 \tag{8}$$

with

$$a_q = \frac{q(q+1)}{R^2}; \; q = 1, 2, \ldots \quad R = \text{radius of the sphere.} \tag{9}$$

Performing the operation ∇_s^2 on both sides of (7) and using (8) one obtains

$$\nabla_s^2 \psi = -\sum_{q=1}^{\infty} a_q \psi_q. \tag{10}$$

Using

$$\int (\nabla_s \psi)^2 \, dF = \int \nabla_s \cdot \psi \nabla_s \psi \, dF - \int \psi \nabla_s^2 \psi \, dF$$

and $\int_F \nabla_s \cdot \psi \nabla_s \psi \, dF = 0$

one obtains

$$\int_F (\nabla_s \psi)^2 \, dF = -\int_F \psi \nabla_s^2 \psi \, dF.$$

Substituting on the right-hand-side of this equation from eqs. (7), (10) and utilizing the orthogonality condition

$$\int_F \psi_q \psi_p \, dF = 0, \; q \neq p$$

one obtains:

$$\int_F (\nabla_s \psi)^2 \, dF = \sum_{q=1}^{\infty} \int_F a_q \psi_q^2 \, dF \tag{11}$$

while on the other hand as is readily seen

$$\int_F (\nabla_s^2 \psi)^2 \, dF = \sum_{q=1}^{\infty} \int_F a_q^2 \psi_q^2 \, dF. \tag{12}$$

Defining H_q from

$$H_q = \int_F a_q \psi_q^2 \, dF$$

one thus gets, combining (11), (12) with (4), (6):

$$\sum_{q=1}^{\infty} H_q = \text{const.} \tag{13}$$

$$\sum_{q=1}^{\infty} a_q H_q = \text{const.} \tag{14}$$

The functions ψ_q divide F into areas F_q where ψ_q has all over either a positive or a negative sign. It is a well established fact that

$$F_q \to 0 \quad \text{when } q \to \infty.$$

The quantity l_q defined from

$$l_q \sim \frac{1}{\sqrt{a_q}} \tag{15}$$

will equal some average diameter of F_q and thus represent a typical scale of the motion which is determined from the streamfunction ψ_q.

If, therefore, H_q is plotted against $\frac{1}{\sqrt{a_q}}$ one has a representation of how the kinetic energy for a given velocity field is distributed over the different components ψ_q with the corresponding scales l_q. The problem to be attacked in the following is to find by the use of the conservation theorems (13), (14) how given initial spectral distributions of energy will change in time, and in particular the more precise upper limits for the flow of energy into components with scales equal to or less than a given one.

The first result to be derived is that when the motion is not a stationary one the kinetic energy must change at least for three different components, or what is the same, on at least three different scales.

To prove this let ΔH_q be defined from

$$\Delta H_q = (Hq)_t - (Hq)_{t=0}.$$

Substituting for the constants in (13), (14), respectively

$$\sum_{q=1}^{\infty} (Hq)_{t=0} \text{ and } \sum_{q=1}^{\infty} a_q (Hq)_{t=0},$$

these equations may be written

$$\sum_{q=1}^{\infty} \Delta H_q = 0 \tag{16}$$

$$\sum_{q=1}^{\infty} a_q \Delta Hq = 0. \tag{17}$$

Let us assume that the changes take place only for the components numbered $q = p$, $q = r$; with $r > p$.

Then (16), (17) reduce to

$$\Delta H_p + \Delta H_r = 0$$
$$a_p \Delta H_p + a_r \Delta H_r = 0.$$

Having $r > p$ it is seen from (9) that the determinant of this system, $a_r - a_p > 0$. Consequently $\Delta H_p = \Delta H_r = 0$. If, however, the changes in kinetic energy take place for three different scales numbered with $q = p$, $q = r$, $q = s$ with $r > p$, $s > r$, the conservation requirements (13), (14) can always be satisfied as seen from the following. In this case (16), (17) reduce to

$$\begin{aligned} \Delta H_p + \Delta H_r + \Delta H_s &= 0 \\ a_p \Delta H_p + a_r \Delta H_r + a_s \Delta H_s &= 0. \end{aligned} \tag{18}$$

Hence

$$\begin{aligned} \Delta H_p &= -\frac{a_s - a_r}{a_s - a_p} \cdot \Delta H_r \\ \Delta H_s &= -\frac{a_r - a_p}{a_s - a_p} \cdot \Delta H_r \end{aligned} \tag{19}$$

Because of $r > p$, $s > r$ and (9) one obtains

$$\frac{a_s - a_r}{a_s - a_p}, \quad \frac{a_r - a_p}{a_s - a_p} > 0.$$

Therefore, in consequence of (19) the change in kinetic energy for the component with the intermediate scale will be opposite of the changes in kinetic energy of the two other components. Accordingly a second result has been obtained which may be formulated as follows: No single of the three components can in this case represent a source or a sink for the *both* two remaining ones unless this is represented by a scale intermediate between the scales of the two other components.

It can also further be understood from

$$\frac{\Delta H_p}{\Delta H_s} = \frac{a_s - a_r}{a_r - a_p} > 1 \tag{20}$$

in connection with (18) that the numerical value of the changes in kinetic energy will be largest for the component with the intermediate scale, and smallest for the one with the smallest scale.

As an example consider

$$a_s/a_r = 4, \quad a_r/a_p = 4$$

so that the corresponding ratio between the different scales are

$$l_s/l_r = 2, \quad l_r/l_p = 2.$$

Then, according to (20),

$$\Delta H_p / \Delta H_s = 4.$$

Therefore, changes in kinetic energy on a certain scale are distributed in the ratio 4/1 on the components with the double and half scales, respectively, if no other components are involved in the energy transformations.

The above result about the nature of the spectral changes for a two-dimensional flow is easily extended to the case where an arbitrary number of components are engaged in the energy transformations.

Writing (16), (17) as

$$\begin{aligned} \sum_{q=1}^{N} \Delta H_q + \sum_{r=N+1}^{N+1+P} \Delta H_r &= 0 \\ \sum_{q=1}^{N} a_q \Delta H_q + \sum_{r=N+1}^{N+1+P} a_r \Delta H_r &= 0 \end{aligned} \tag{21}$$

and assuming

$$\Delta H_q \leqq 0; \; q = 1, 2 \ldots N$$
$$\Delta H_r \geqq 0; \; r = N+1, \ldots N+1+P, \tag{22}$$

(21) may also be written

$$\begin{aligned} \sum_{q=1}^{N} \Delta H_q + \sum_{r=N+1}^{N+1+P} \Delta H_r &= 0 \\ a^{\star} \sum_{q=1}^{N} \Delta H_q + a^{\star\star} \sum_{r=N+1}^{N+1+P} \Delta H_r &= 0 \end{aligned} \tag{23}$$

where now because of (9) and (22)

$$a_1 \leqq a^{\star} \leqq a_N$$
$$a_{N+1} \leqq a^{\star\star} \leqq a_{N+1+P}.$$

Hence, consulting (9), the determinant of the system (23) $a^{\star\star} - a^{\star} > 0$.

The assumption (22) is therefore not possible.

A problem of considerable interest now presents itself in connection with the determination of an upper limit to the flow of energy into components having scales equal to or less than a given one.

We introduce the notations h and h_q defined from

$$(H_q)_{t=0} = h_q; \; h = \sum_{q=1}^{\infty} h_q,$$

whereas a, $a^{\star}$, and $a^{\star\star}$ are defined from the equations:

$$\sum_{q=1}^{\infty} a_q h_q = a \sum_{q=1}^{\infty} h_q \tag{24}$$

$$\sum_{q=1}^{N} a_q H_q = a^{\star} \sum_{q=1}^{N} H_q \tag{25}$$

$$\sum_{q=N+1}^{\infty} a_q H_q = a^{\star\star} \sum_{q=N+1}^{\infty} H_q. \qquad (26)$$

Eqs. (23) may then be written formally as the system of equations

$$\sum_{q=1}^{N} H_q + \sum_{q=N+1}^{\infty} H_q = \sum_{q=1}^{\infty} h_q$$

$$a^{\star} \sum_{q=1}^{N} H_q + a^{\star\star} \sum_{q=N+1}^{\infty} H_q = a \sum_{q=1}^{\infty} h_q$$

From a formal solution of these equations one obtains

$$\sum_{q=N+1}^{\infty} H_q : \sum_{q=1}^{N} H_q = \frac{a - a^{\star}}{a^{\star\star} - a}. \qquad (27)$$

H_q, h_q being quantities which are all positive or zero, and a_q increasing monotonically with q, it follows from (24), (25), (26) that

$$a_1 \leqq a \qquad (28)$$

$$a_1 \leqq a^{\star} \leqq a_N \qquad (29)$$

$$a_{N+1} \leqq a^{\star\star}. \qquad (30)$$

Accordingly

$$a^{\star\star} > a^{\star}.$$

This, in connection with the fact that the ratio on the left-hand-side of (27) is positive or zero fixes $a^{\star}$ and $a^{\star\star}$ to assume values which are related to the given value of a as

$$a^{\star} \leqq a \leqq a^{\star\star}. \qquad (31)$$

For the ratio between the kinetic energy contained within the range $q \gtreqless N+1$, and the total energy, we get

$$\sum_{q=N+1}^{\infty} H_q : h = \frac{a - a^{\star}}{a^{\star\star} - a^{\star}}. \qquad (32).$$

The maximum value of this ratio is assumed when $a^{\star\star} = a$. This is, as seen from (31), (30), only possible if

$$a_{N+1} \leqq a. \qquad (33)$$

Before we proceed to the most interesting case with

$$a_{N+1} > a, \qquad (34)$$

we will discuss the first case in connection with the problem of stability of the stationary motion [1])

$$\psi = \psi_1. \qquad (35)$$

ψ_1 must not necessarily be the "eigen"-function with the lowest "eigen"-value for the *spherical* "eigen"-valueproblem, (8), but may be the "eigen"-function with the lowest "eigen"-value of the most general twodimensional "eigen"-value problem

$$\nabla_2^2 \psi_q + a_q \psi_q = 0$$

$$\psi_q = 0 \text{ at a boundary } L,$$

yielding

$$\begin{cases} a_{q=1} > a_q, \ a_q > 0; \ q = 1, \ldots \\ a_q \to \infty, \ q \to \infty. \end{cases} \qquad (36)$$

When instead of (35),

$$\psi_{t=0} = \psi_1 + \sum_{q=2}^{\infty} \psi_q'$$

we may write

$$h = h_1 + \sum_{q=2}^{\infty} H_q'$$

$$ah = a_1 h_1 + \sum_{q=2}^{\infty} a_q h_q'$$

yielding

$$a = \frac{a_1 h_1 + \sum_{q=2}^{\infty} a_q h_q'}{h_1 + \sum_{q=2}^{\infty} a_q h_q'}. \qquad (37)$$

At any later time we have now according to (32), when N is put equal to 1:

$$\sum_{q=2}^{\infty} H_q' : h = \frac{a - a^{\star}}{a^{\star\star} - a^{\star}}.$$

Now, with $N = 1$, eqs. (28), (30) become

$$\begin{aligned} a_1 &= a^{\star} \\ a_2 &\leqq a^{\star\star}. \end{aligned} \qquad (38)$$

Accordingly

$$\sum_{q=2}^{\infty} H_q : h = \frac{a - a_1}{a^{\star\star} - a_1}. \qquad (39)$$

$\sum_{q=2}^{\infty} a_q h_q'$ represents the sum of the squares of the vorticities of the initial disturbance. Let us assume that

$$\sum_{q=2}^{\infty} a_q h_q' \to 0. \qquad (40)$$

Because of (36) it follows then necessarily that also

$$\sum_{q=2}^{\infty} h_q' \to 0. \qquad (41)$$

Using (40) and (41) in (37) one obtains

$$a \to a_1, \text{ when } \sum_{q=2}^{\infty} a_q h_q' \to 0.$$

[1]) The stationarity follows from the fact that $\psi = \psi_1$ satisfies eq. (5) with $\frac{\partial \nabla_s^2 \psi}{\partial t} = 0$.

Therefore it follows from (39) in connection with (38)

$$\sum_{q=2}^{\infty} H_q : h \to 0 \text{ when } \sum_{q=2}^{\infty} a_q h_q \to 0. \quad (42)$$

This expresses the result that the considered stationary motion $\psi = \psi_1$ is stable in the sense that the total kinetic energy of the disturbance, $\sum_{q=2}^{\infty} H_q$, at all times is kept below a limit which goes to zero if the sum of the squares of the vorticities of the initial disturbance goes to zero.

Let us now on the other hand assume that

$$\sum_{q=2}^{\infty} h'_q = \varepsilon.$$

By concentrating this energy on sufficiently small scales it is possible because of (36) to make

$$\sum_{q=2}^{\infty} a_q h'_q > P,$$

where P may be chosen arbitrarily large, however small ε is taken. Consulting the expression for a in (37) it is therefore understood that it is always possible, however small ε is taken, to make a arbitrarily large. Particularly a may be made equal to or larger than a_2 which accordingly to (38) is a sufficient condition for the possibility of getting $a^{\star\star} = a$, and then also to obtain

$$\sum_{q=2}^{\infty} H_q : h = 1,$$

however small values are taken for the initial kinetic energy of the disturbance.

From this example it is seen how important it is for the *proof* of stability that the disturbance be assumed to be small also as regards its vorticities, and not only with regard to its energy, even though it is still questionable whether the stationary flow *actually* will behave unstably if this is not the case.

The basic flow $\psi = \psi_1$ represents the component with the largest possible scale. The stability of this flow may therefore also be interpreted as follows: A necessary condition for instability of a stationary motion for two-dimensional flow is that the disturbances be represented also by components having scales which are larger than the scale of the basic flow. Thus interpreted, the necessary condition for instability becomes directly connected with the more general results found for the changes in the spectral distribution of energy. It is also further easily extended to flows where the boundary condition is not necessarily $\psi = \text{const}$, as for instance a flow between parallel walls. Taking this as granted, we may apply the result to a linear flow represented by a streamfunction $\text{const} \cdot \cos \frac{2\pi y}{L^\star}; 0 \leq y \leq H$. The scale of this basic flow is determined from the corresponding »eigen»-value $\frac{4\pi^2}{L^{\star 2}}$ whereas the scales of the components of the disturbances are determined from the »eigen»-values $\frac{4\pi^2}{L^2} + \frac{4\pi^2 q^2}{(2H)^2}$ when a wavelength L is supposed for the dependence upon the direction along the walls. Thus the linear flow considered will possibly be unstable only if

$$\frac{4\pi^2}{L^2} + \frac{4\pi^2}{(2H)^2} < \frac{4\pi^2}{L^{\star 2}}. \quad (43)$$

We now proceed to the case (34). The upper limit of the ratio in (32) is then reached when $a^\star$ and $a^{\star\star}$ assume their smallest possible values, which according to (29), (30) are a_1 and a_{N+1}, respectively. Accordingly

$$\sum_{q=N+1}^{\infty} H_q : h \leq \frac{a - a_1}{a_{N+1} - a_1} = \frac{a}{a_{N+1}} \cdot \frac{1 - \frac{a_1}{a}}{1 - \frac{a_1}{a_{N+1}}} < \frac{a}{a_{N+1}}. \quad (44)$$

To apply (44) to a seemingly important case, assume

$$h = \sum_{q=1}^{L} h_q.$$

The quantity a defined from (24) will now lay between the limits

$$a_1 \leq a \leq a_L. \quad (45)$$

To ensure (34) it is therefore sufficient to assume $L < N+1$. We may now write (44)

$$\sum_{q=N+1}^{\infty} H_q : h \leq \frac{a - a_1}{a_{N+1} - a_1} \leq \frac{a_L - a_1}{a_{N+1} - a_1} = \left(\frac{l_{N+1}}{l_L}\right)^2 \cdot \frac{\left(1 - \frac{l_L}{l_1}\right)^2}{\left(1 - \frac{l_{N+1}}{l_1}\right)^2}$$

having introduced the scales defined in (15). In the considered case, therefore, the fraction of the total energy which can flow to components with scales equal to or less than a certain scale l_{N+1} is less than the square of the ratio between the smallest scale l_L, represented initially, and l_{N+1}. If therefore the initial flow is represented by a typical large-scale velocity field (L = relatively small), the flow of kinetic energy into the smaller eddies will rapidly become unimportant.

Obviously it is not necessary to assume $h_q = 0$ for all $q > L$ to obtain this result. It is sufficient only to have

$$a \leqq a_L$$

and L sufficiently small to get the corresponding field to be characterized as a "typical" large-scale-field.

2. Twodimensional flow of a viscous fluid

With molecular friction the equations governing the motion become

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla_s \gamma - (\mathbf{v}\cdot\nabla_s \mathbf{v})_s + \nu\,\nabla_s^2 \mathbf{v}$$

$$\mathbf{v} = -\nabla_s \psi \times \mathbf{k}.$$

Eliminating $\nabla_s \gamma$ one obtains the vorticity equation

$$\frac{\partial \nabla_s^2 \psi}{\partial t} = -\mathbf{v}\cdot\nabla_s \nabla_s^2 \psi + \nu\,\nabla_s^4 \psi.$$

The changes per unit time of total kinetic energy and total vorticity squared now become

$$\frac{d}{dt}\int_F (\nabla_s \psi)^2\, dF = -\nu \int_F (\nabla_s^2 \psi)^2\, dF$$

$$\frac{d}{dt}\int_F (\nabla_s^2 \psi)^2\, dF = \nu \int_F \nabla_s^2 \psi \cdot \nabla_s^4 \psi\, dF.$$

Making use of (8) and (9) one may write these equations as

$$\frac{d}{dt}\int_F (\nabla_s \psi)^2 dF = -\nu A \int_F (\nabla_s \psi)^2 dF;\ A > 0. \quad (46)$$

$$\frac{d}{dt}\int_F (\nabla_s^2 \psi)^2 = -\nu B \int_F (\nabla_s^2 \psi)^2\, dF;\ B > 0. \quad (47)$$

Here

$$B(t) \leqq A, \quad (48)$$

where the equality sign is to be taken only when ψ is represented by a single component, $\psi = \psi_q$. This follows easily from the defining equations for A and B

$$\sum_{q=1}^{\infty} a_q H_q = A \sum_{q=1}^{\infty} H_q$$

$$\sum_{q=1}^{\infty} a_q^2 H_q = B \sum_{q=1}^{\infty} a_q H_q$$

together with (9). From (46), (47), (48) one now gets:

$$\sum_{q=1}^{\infty} H_q : \sum_{q=1}^{\infty} a_q H_q = n\,[h : ah] = \frac{n}{a}$$

where

$$n \leqq 1 \text{ according as } B \leqq A. \quad (49)$$

It is now easily demonstrated that eq. (27) now has to be written

$$\sum_{q=N+1}^{\infty} H_q : \sum_{q=1}^{N} H_q = \frac{\frac{a}{n} - a^{\star}}{a^{\star\star} - \frac{a}{n}}.$$

Since now the upper limit to this ratio decreases with decreasing $\frac{a}{n}$, apart from the steady decrease in kinetic energy and vorticity because of friction, a flow of energy to smaller scales should so far (because $n > 1$) have a smaller chance to be realized than for the inviscid fluid, for which $a/n = a < a/n$.

3. Conclusion.

The nature of the changes in the spectral distribution of kinetic energy in a twodimensional flow differ radically from the changes taking place when real turbulence develops. It is natural to believe that this discrepancy is due to the fact that for the development of real turbulence it is essential to consider the motion in three dimensions in which case as is well known no conservation requirement regarding the square of vorticities has to be fulfilled.

The implications of (43) and its relation to earlier works of the author and others will be discussed more thoroughly in an article to appear later.