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The thermocline problem

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1. Introduction

The main oceanic thermocline, the layer of strong vertical temperature and salinity gradients in the ocean, and the region below this, where the water asymptotically approaches the deep-sea state, is still far from explained. There have appeared a series of theoretical studies of this phenomenon in recent years, starting with a linearized model by P. S. Lineykin and later concentrating on the nonlinear problem retaining the important nonlinear density advection. A summary of the work up to 1969 has been given by Veronis (1969).

It must be admitted that we are still uncertain even about the basic mechanism of the main thermocline. There appears to be several reasons for such an uncertainty. First, we have few direct measurements of horizontal velocities, and none of vertical velocity and vertical turbulent diffusion of heat and salt in the thermocline. The vertical velocities are so small, of order 10^{-5} cm/s that no known instrument can feel them.

The diffusion coefficients could possibly be measured, by releasing a tracer in the thermocline and measuring the resulting concentration in space-time. Such an experiment would be a major operation, and certainly not an easy one. So far, it has not been attempted. Estimates of the vertical diffusion coefficient have been made from profiles of heat, salt, stable geochemical tracers and radioactive tracers, such as ^{14}C, assuming that some quasi balance of vertical advection and diffusion exists. This is not, however, a completely fair method. Certainly, we should not be impressed if, in our theoretical thermocline models, the vertical profiles come out in good agreement with nature if the diffusion coefficient used has itself been determined from observed profiles.

We should, in principle, be able to deduce the diffusion coefficient from theoretical arguments. We should also be able to produce its dependence on the Richardson number, etc. As is well known, turbulence theory has not advanced so far. We further fear that the problem of self-generated turbulence in a stratified shear flow is not the relevant one. Internal waves, generated at other locations, could be the source of instability, by causing unstable shear-layers or by direct breaking. It is possible that the so-called 'salt-finger' mechanism, described by Stern (1960), is the main diffusing agency in the thermocline. Then we have a completely new situation even energetically, as this process releases available potential energy in the mass field.

Finally, there is the possibility that all diffusing agencies are unimportant in the main thermocline, or at least part of it, and that the observed structure results from an ideal fluid advection. One would think that the two alternatives of (*a*) mixing being a dominant factor, and (*b*) mixing being unimportant, could be tested theoretically. Unfortunately, the theoretical solutions that we have considered so far have failed in this task. Peculiarly enough, the known exact solutions to the thermocline equations including the vertical diffusion (with a constant diffusivity) are also

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solutions to the ideal fluid problem! One only needs to make a slight change in the boundary conditions, applied below a top Ekman layer, to transform one problem into the other.

Clearly, the difficulty of theoretical testing hangs on the fact that we do solve only a partial problem. The solutions we have found for the nonlinear thermocline problem are of similarity type (some can be worked out exactly, others can be calculated numerically from an ordinary differential equation). The similarity restriction is, of course, felt also in the boundary conditions. These solutions cannot satisfy the arbitrary conditions we want to impose on the top and bottom of the ocean. It seems likely that the solutions that we restrict our attention to are not 'typical solutions'. They actually represent a very particular balance of terms. This certainly simplifies our mathematics. However, this may not represent nature's way of balancing!

To come to grips with the problem, a fresh start seems needed. Attempts to obtain more general solutions by series expansion methods have been made by P. S. Lineykin (unpublished), and the thermocline problem has also been modelled numerically by Bryan & Cox (1968). However, the expansion method is not yet fully explored, nor have the numerical calculations been sufficient to give a definite answer (for one thing, the grids are so coarse in these calculations that details in the boundary layer cannot be well resolved).

Below is presented a formulation of the problem that reveals some novel features. In particular, the necessity for an ideal fluid régime is shown, for values of the vertical diffusivities below $1\,\mathrm{cm^2\,s^{-1}}$. The governing equation cannot be solved analytically, subject to general boundary conditions, but solutions can be generated effectively by a method of successive linearization. From such calculations it should be possible to demonstrate conclusively the different régimes which appear in the thermocline problem.

2. The thermocline equations and the M-equation

It will be assumed, as is usual, that the Earth is spherical with a spherical-symmetric field of (apparent) gravity, that the ocean characteristic depth H is small compared to the radius of Earth R, and to the horizontal scales L involved. The Boussinesq approximation is made, and it is assumed that the vertical diffusion of heat and salt are characterized by the same constant eddy coefficient. It is then possible to translate the salinity effect into an equivalent temperature effect.

Finally, it is assumed that the Rossby number is small enough to justify the geostrophic balance equations in the horizontal, and that the Ekman depth is small compared to the thermal boundary layer depths that may appear under these conditions. The steady-state equations, away from frictional top and bottom layers and away from eventual side boundaries are then:

$$-fv = -\frac{1}{R\cos\phi}P_\lambda, \quad fu = -\frac{1}{R}P_\phi, \quad P_z = g\alpha T, \tag{1, 2, 3}$$

$$\frac{1}{R\cos\phi}\{\phi_\lambda + (v\cos\phi)_\phi\} + w_z = 0, \tag{4}$$

$$\frac{u}{R\cos\phi}T_\lambda + \frac{v}{R}T_\phi + wT_z = \kappa T_{zz}, \tag{5}$$

λ, ϕ, z are longitude, latitude and vertical distance upward, measured from a horizontal surface at the bottom of the top Ekman layer. (u, v, w) is the velocity, P perturbation pressure (divided by ρ), and T a perturbation temperature. g is the acceleration of gravity, α the thermal expansion coefficient, κ the vertical diffusivity, and $f = 2\Omega \sin\phi$ the Coriolis parameter.

Solving u, v, T in terms of P from (1, 2, 3), and then w in terms of P from (5), and inserting these expressions into (4) gives a single nonlinear equation for the pressure, derived earlier by Needler (1967). It is of the fourth order and second degree (products of the highest derivatives occur). The following derivation gives a simpler equation. Define the function:

$$M(\lambda, \phi, z) = \int_0^z P\,\mathrm{d}z + 2\Omega R^2 \sin^2\phi \int_0^\lambda w_0\,\mathrm{d}\lambda, \tag{6}$$

where $w_0 = w(\lambda, \phi, 0)$. Then

$$u = -\frac{1}{2\Omega R \sin\phi} M_{\phi z}, \quad v = \frac{1}{2\Omega R \sin\phi \cos\phi} M_{\lambda z},$$

$$w = \frac{1}{2\Omega R^2 \sin^2\phi} M_\lambda, \quad P = M_z, \quad T = \frac{1}{g\alpha} M_{zz},$$

where w is obtained from combining (1, 2, 3) and integrating after z. Inserting these expressions into (5) gives the M-equation:

$$2\Omega R^2 \kappa \sin\phi \cos\phi M_{zzzz} + \frac{\partial(M_{zz}, M_z)}{\partial(\lambda, \phi)} - \cot\phi M_\lambda M_{zzz} = 0. \tag{7}$$

This is similar to the integrated pressure equation used earlier (Welander 1959; Robinson & Welander 1963), but it is more general, since there is no assumption made about the behaviour of the solution at great depth. The function w_0 that appears in the definition of M is prescribed, this is the vertical velocity produced at the bottom of the top Ekman layer by the wind-stress. One boundary condition in M is thus:

$$M = 2\Omega R^2 \sin^2\phi \int_0^\lambda w_0\,\mathrm{d}\lambda \quad \text{at} \quad z = 0. \tag{8}$$

Further, one must have a thermal condition specified at the top. This could be temperature, heat flux or a relation between heat flux and a difference in air–sea temperature. Taking the simplest case, with a prescribed temperature, we have

$$M_{zz} = g\alpha T_0(\lambda, \phi). \tag{9}$$

At the bottom of the ocean, $z = -H(\lambda, \phi)$, we require that the normal velocity and heat flux vanish (there will, of course, be a frictional boundary layer also here, but this is neglected in the interior ocean, the case considered here. In a complete model, including side boundaries, it must be kept). One usually requires that the bottom is thermally insulating. However, as it turns out, a more natural thermal condition is to put the bottom temperature at a constant value, say, $T = 0$.† Then we have

$$\cot\phi\, M_\lambda = \frac{\partial(H, M_z)}{\partial(\lambda, \phi)}, \quad M_{zz} = 0. \tag{10, 11}$$

Scale these equations, for convenience, according to the following transformation:

$$H \to \bar{H}.H, \quad z \to \delta_a.z, \quad P \to (2\Omega R^2 W g\alpha\mathscr{T})^{\frac{1}{2}}.P,$$

$$(u, v) \to \left(\frac{Wg\alpha\mathscr{T}}{2\Omega}\right)^{\frac{1}{2}}.(u, v), \quad w \to W.w, \quad T \to \mathscr{T}.T, \quad M \to 2\Omega R^2 W.M,$$

† The explanation for this is involved and will not be discussed here. In short, the insulating condition will not give enough formation of deep water. At least in thermocline models with constant κ, and driven by time-independent heating, some 'faking' with the bottom condition seems needed to obtain a realistic deep circulation.

where W and $\mathscr{T}$ are the amplitudes of w_0 and T_0, $\bar{H}$ is a mean ocean depth, and

$$\delta_a = (2\Omega R^2 W/g\alpha\mathscr{T})^{\frac{1}{2}},$$

a characteristic (advective) depth.

Then the equations (1, 2, 3, 4) contain no parameters. A parameter, $\epsilon = \delta_d/\delta_a$, with $\delta_d = \kappa/W$, a second characteristic (diffusive) depth, appears in the diffusion term. The problem in terms of the scaled M takes on the form:

$$\epsilon \sin\phi \cos\phi\, M_{zzzz} + \frac{\partial(M_{zz}, M_z)}{\partial(\lambda, \phi)} - \cot\phi\, M_\lambda M_{zzz} = 0 \tag{12}$$

with boundary conditions

$$M = \sin^2\phi \int_0^\lambda w_0 \,\mathrm{d}\lambda, \quad M_{zz} = T_0 \quad \text{at} \quad z = 0, \tag{13, 14}$$

$$\gamma \cot\phi\, M_\lambda = \frac{\partial(H, M_z)}{\partial(\lambda, \phi)}, \quad M_{zz} = 0 \quad \text{at} \quad z = -\frac{1}{\gamma}H, \tag{15, 16}$$

where γ is a second parameter, $\gamma = \delta_a/\bar{H}$. This constitutes the mathematical problem. The purpose is to discuss this on a sphere, or part of a sphere, without introducing the complications of side boundaries. Horizontal scales are assumed to be everywhere given by the forcing functions, and of order R in the dimensional equations.

3. Régimes of the M-equation

The previous scaling represented only a formal transformation, with no assumptions introduced. The scaling demonstrating the different possible régimes of the equation requires, however, certain assumptions that, of course, should be verified *a posteriori*.

In the vertical direction an unknown length-scale δ is introduced. It is assumed that the horizontal velocity (u, v) can have a barotropic component $(u^{\mathrm{b}}, v^{\mathrm{b}})$ (independent of depth), that is at most of order unity in the scaled equations. (This means, it is at most of order of the baroclinic velocity, calculated over the advective depth δ_a.)

When estimating M_z we must accordingly write:

$$M_z \approx \Delta M/\delta + M_z^{\mathrm{b}}, \quad M_z^{\mathrm{b}} \lesssim 1.$$

Now, starting at the top let us examine the régimes. The top boundary conditions require:

$$M \approx 1, \quad \frac{\Delta M}{\delta^2} \approx 1.$$

In the M equation, it will be necessary to keep the ϵM_{zzzz}-term to meet the diffusive boundary condition. Two possible balance conditions exist (one cannot obviously have the M_{zzzz}-term alone):

(i) ϵM_{zzzz} and $[\partial(M_{zz}, M_z)]/[\partial(\lambda, \phi)]$ are of the same order. Then:

$$\epsilon \frac{\Delta M}{\delta^4} \approx \frac{\Delta M}{\delta^2}\left[\frac{\Delta M}{\delta} + M_z^{\mathrm{b}}\right].$$

It follows: $\quad \epsilon/\delta^2 \approx \delta + M_z^{\mathrm{b}}, \quad \epsilon^{\frac{1}{2}} \ll \delta \ll \epsilon^{\frac{1}{3}}, \quad \epsilon \ll \Delta M \ll \epsilon^{\frac{2}{3}}.$

But then

$$\frac{M_\lambda M_{zzz}}{\dfrac{\partial(M_{zz}, M_z)}{\partial(\lambda, \phi)}} \approx \frac{1 \,.\, \Delta M/\delta^3}{\dfrac{\Delta M}{\delta^2}\left(\dfrac{\Delta M}{\delta} + M_z^{\mathrm{b}}\right)} \approx \frac{1}{\delta(\delta + M_z^{\mathrm{b}})} \gg 1$$

and the case is inconsistent. (It should be noted that our scaling excludes the pole and equator. We always assume $\sin\phi \approx \cos\phi \approx 1$. The other cases have to be looked at separately.)

(ii) ϵM_{zzzz} and $M_\lambda M_{zzz}$ are of the same order. Then:

$$\epsilon\frac{\Delta M}{\delta^4} \approx 1.\frac{\Delta M}{\delta^3} \quad \text{and} \quad \delta \approx \epsilon,\ \Delta M \approx \epsilon^2$$

Now

$$\frac{\dfrac{\partial(M_{zz}, M_z)}{\partial(\lambda, \phi)}}{M_\lambda M_{zzz}} \sim \frac{\dfrac{\Delta M}{\delta^2}\left(\dfrac{\Delta M}{\delta}+M_z^{\mathrm{b}}\right)}{1.\Delta M/\delta^3} \sim \delta(\delta+M_z^{\mathrm{b}}) \ll 1.$$

This case is thus consistent.

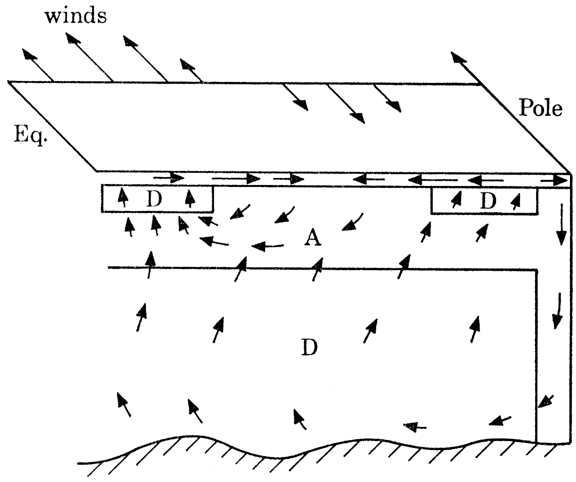

FIGURE 1. A schematic picture of the circulation and the thermocline régimes in an ocean driven by wind-stress and differential heating. The section is along a mid-oceanic longitude. The western boundary currents may represent an additional diffusive régime. D, diffusive régime; A, advective régime.

The third case where all three terms ϵM_{zzzz}, $[\partial(M_{zz}, M_z)]/[\partial(\lambda, \phi)]$, and $M_\lambda M_{zzz}$ are of the same order is, of course, inconsistent.

In the régime found, the scale depth is ϵ (δ_d in dimensional variables). M does not vary essentially in vertical direction and can be replaced by the value at the boundary. The M equation reduces to $\epsilon M_{zzzz} = w_0 M_{zzz}$, with the solution $M = a+bz+cz^2+d\mathrm{e}^{w_0 z/\epsilon}$. The boundary conditions give

$$a+d = \sin^2\phi\int_0^\lambda w_0\,\mathrm{d}\lambda,\ 2c+d(w_0/\epsilon)^2 = T_0.$$

Further, when $w_0 < 0$ one must choose $d = 0$. This régime represents a diffusive boundary layer, with heat diffusion balancing an upward Ekman vertical velocity. When the vertical velocity is downward, it just gives vertically constant temperature.

Below this diffusive régime, the vertical velocity is of the same order as at the top. There will still be horizontal temperature variations of the same order as at the surface (this follows already from the fact that the temperature is vertically unchanged in the downwelling regions). Thus, we must have another thermal boundary layer below that can bring the temperature down to the deep constant value.

In this boundary layer we should have

$$M \approx 1, \quad \Delta M/\delta^2 \approx 1,$$

Which new régime can be found consistent with such conditions? The only possible new balance is between $[\partial(M_{zz}, M_z)]/[\partial(\lambda, \phi)]$ and $M_\lambda M_{zzz}$. (Since we are now away from the boundary, the diffusive term must not necessarily be included.)

This gives
$$\frac{\Delta M}{\delta^2}\left(\frac{\Delta M}{\delta}+M_z^{\mathrm{b}}\right) \approx 1.\frac{\Delta M}{\delta^3} \quad \text{or} \quad \delta + M_z^{\mathrm{b}} \approx 1/\delta.$$

Since $M_z^{\mathrm{b}} \lessapprox 1$, this means that $\delta \approx 1$. Further $\Delta M \approx 1$ and $\epsilon M_{zzzz}/M_\lambda M_{zzz} \approx \epsilon \ll 1$, making this ideal fluid régime consistent. Note that the dimensional scale is the advective depth δ_a introduced earlier.

The need for this régime can be seen in another way, that may be more physical. Returning to the dimensional equations, we have for the 'classical' thermocline problem the following balance requirements, across a thermocline of thickness D, set up by temperature variations of order ΔT (Robinson 1960):

$$\frac{f_0 V}{D} \approx \frac{g\alpha \Delta T}{R} \quad \text{(thermal wind relation)},$$

$$\frac{V}{R} \approx \frac{W}{D} \quad \text{(continuity)},$$

$$\frac{W\Delta T}{D} \approx \frac{\kappa \Delta T}{D^2} \quad \text{(balance of vertical advection and diffusion)}.$$

The horizontal scale is again assumed of order R. These conditions determine D and the characteristic horizontal and vertical velocities V and W:

$$D \approx \kappa^{\frac{1}{3}}\left(\frac{f_0 R^2}{g\alpha\Delta T}\right)^{\frac{1}{3}}, \quad V \approx R\kappa^{\frac{1}{3}}\left(\frac{g\alpha\Delta T}{f_0 R^2}\right)^{\frac{2}{3}},$$
$$W \approx \kappa^{\frac{2}{3}}\left(\frac{g\alpha\Delta T}{f_0 R^2}\right)^{\frac{1}{3}}.$$

In terms of the diffusive and advective depths δ_d and δ_a used earlier, we find $D \sim \delta_d^{\frac{1}{3}}\delta_a^{\frac{2}{3}}$. Therefore, this thermocline has an intermediate scale depth. Further one finds that W is of order $\epsilon^{\frac{2}{3}}$ relative to the prescribed Ekman vertical velocity. *The classical thermocline scaling cannot be used because the resulting vertical velocity, internally determined, is too small to match the prescribed Ekman vertical velocity.* This régime can only be applied after the vertical velocity value has been reduced. The top diffusive layer is of no help, but an ideal fluid régime must be placed in between.

What actually happens at the bottom of the ideal thermocline is still not clear. Will the velocities and temperature decay in such a way to keep the diffusive terms small (except in a bottom boundary layer), or will diffusion again become a dominant term? Studies of some exact. solutions (Welander 1971), indicates that the diffusive term will come in again in a deep régime It is not known, however, how general this result is.

In conclusion, this discussion suggests a picture reversed from the one commonly assumed. The thermocline may not be a diffusive boundary layer but rather an ideal fluid régime imbedded between diffusive régimes.

Of course, in the application to the ocean, all hangs on the value of κ. If one assumes that $\kappa = 1\,\mathrm{cm^2\,s^{-1}}$, and sets $R = 6 \times 10^8\,\mathrm{cm}$, $f = 10^{-4}\,\mathrm{s^{-1}}$, $g = 10^3\,\mathrm{cm\,s^{-2}}$, $\alpha\mathscr{T} = 10^{-3}$, $W = 10^{-4}\,\mathrm{cm\,s^{-1}}$, the result is $\delta_d \approx 100\,\mathrm{m}$, $\delta_a \approx 600\,\mathrm{m}$.

In this case, the ideal fluid régime should be seen, but the different régimes are not strongly separated. Decreasing κ to $0.1\,\mathrm{cm^2\,s^{-1}}$ gives $\delta_d \approx 10\,\mathrm{m}$, $\delta_a \approx 600\,\mathrm{m}$, $\epsilon \approx \frac{1}{60}$. In this case, the ideal fluid thermocline should stand out very clearly.

In the numerical experiments by Bryan & Cox (1968), the separation is not strong and, as has been said, the boundary layers are not resolved in any detail. Still in his picture for the balance of terms in the heat transport equation, at subtropical latitudes, signs of an ideal fluid régime can be traced. (See Bryan & Cox, Fig. 4*b*, p. 972.)

4. Numerical treatment of the M-equation

The solution to the boundary-value problem is considered for a region on the spherical Earth, between two latitudes $\phi = \phi_0$ and ϕ_1, which are considered as slip-boundaries. $w_0(\lambda, \phi)$, $T_0(\lambda, \phi)$, or $T_z(\lambda, \phi)$, are prescribed at the top, care being taken that total mass and heat flux are balanced. The forcing functions and the bottom topography function $H(\lambda, \phi)$ are further chosen such that no side boundary layers are induced. It is necessary to go to a numerical technique to solve this problem, in a general case. Time-stepping has been attempted, adding a term M_{zzt} to (12). (This represents the local rate of change of temperature in a time-dependent, geostrophic thermocline model.) The method is not good, because the numerical stability requires small time steps compared to the overall relaxation time. An iteration of the M equation in the form

$$\epsilon \sin\phi \cos\phi M_{zzzz}^{(n+1)} + \frac{\partial(M_{zz}^{(n)}, M_z^{(n)})}{\partial(\lambda, \phi)} - \cot\phi M_\lambda^{(n)} M_{zzz}^{(n+1)} = 0$$

has also been attempted. A similar method was used by Stommel & Webster (1962) for a simplified thermocline model. The method is convergent, at least for a certain range of ϵ-values, but the convergence is slow in the present case. The best method found so far is a successive linearization, starting from an approximation $M_0 = A(\lambda, \phi) + B(\lambda, \phi)\, z + C(\lambda, \phi)\, e^{\kappa z/\sin\phi}$. This is actually an exact solution to (12), with A and C arbitrary functions and k an arbitrary constant. B is related to A and C by a first-order differential equation. This solution, given by Needler (1967), and earlier by Welander (1959) for the ideal fluid case, is the most general exact solution for the thermocline equation found so far.

In numerical procedure A, B and C are determined from the boundary conditions (13, 14, 15). The condition (16) is approximately satisfied when the ocean is deep enough. The value of k is chosen to be unity (possibly a different value would give a better convergence). The second approximation is obtained by writing $M = M_0 + M_1$ and linearizing around M_0, etc. The boundary conditions are satisfied for each approximation. Already in the second approximation, all three régimes mentioned earlier appear. The results of these numerical experiments, which are not yet completed, will be published elsewhere.

References (Welander)

Bryan, K. & Cox, M. 1968 A non-linear model of the ocean driven by wind and differential heating. Part I and II. *J. atmos. Sci.* **25**, 945–978.

Needler, G. T. 1967 A model for the thermohaline circulation in an ocean of finite depth. *J. mar. Res.* **25**, 329–342.

Robinson, R. 1960 The general thermal circulation in equatorial regions. *Deep Sea Res.* **6**, 311–317.

Robinson, R. & Welander, P. 1963 Thermal circulation on a rotating sphere, with application to the oceanic thermocline. *J. mar. Res.* **21**, 25–38.

Stern, M. 1960 The salt fountain and the thermocline convection. *Tellus* **12**, 172–175.

Stommel, H. & Webster, F. 1962 Some properties of the thermocline equations in a subtropical gyre. *J. mar. Res.* **20**, 42–56.

Veronis, G. 1969 On theoretical models of the thermohaline circulation. *Deep. Sea Res.* **16** Suppl. 301–323.

Welander, P. 1959 An advective model of the ocean thermocline. *Tellus* **11**, 309–318.

Welander, P. 1971 Some exact solutions to the equations describing an ideal-fluid thermocline. *J. mar. Res.* **29**, 60–68.