Fluid Dynamics of the Atmosphere and Oceans

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Introduction

This module Fluid Dynamics of the Atmosphere and Oceans comprises 33 hours of lectures and examples classes.

Outline of course content

The equations of motion in a rotating frame; some conservation properties; circulation theorem; vorticity equation; potential vorticity equation. Hierarchies of approximate governing equations; balance and filtering.

Shallow water equations: circulation and potential vorticity; energy and angular momentum; gravity and Rossby waves; geostrophic balance; geostrophic adjustment; Rossby radius; quasigeostrophic shallow water equations; quasigeostrophic potential vorticity; Rossby waves; Kelvin waves.

Boussinesq approximation; gravity waves in three dimensions; mountain waves; nonlinear effects; eddy fluxes.

Shallow atmosphere hydrostatic primitive equations, in different coordinate systems; conservation properties; Rossby and gravity waves.

Quasigeostrophic theory in three dimensions: ageostrophic equations; quasigeostrophic potential vorticity equation; omega equation; free Rossby waves; Forced Rossby waves and the Charney Drazin theorem; eddy fluxes; surface waves on a potential temperature gradient; the Eady model of baroclinic instability.

The planetary boundary layer; the Ekman spiral; Ekman pumping; Sverdrup balance.

1 Governing equations in vector form

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi; \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0; \tag{2}$$

$$c_v \frac{DT}{Dt} + \frac{p}{\rho}\nabla \cdot \mathbf{u} = 0. \tag{3}$$

2 Governing equations in spherical polar coordinates

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} - 2\Omega v \sin\phi + 2\Omega w \cos\phi + \frac{1}{\rho r \cos\phi}\frac{\partial p}{\partial \lambda} = 0 \tag{4}$$

$$\frac{Dv}{Dt} + \frac{u^2 \tan\phi}{r} + \frac{vw}{r} + 2\Omega u \sin\phi + \frac{1}{\rho r}\frac{\partial p}{\partial \phi} = 0 \tag{5}$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u \cos\phi + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{6}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{7}$$

$$c_v \frac{DT}{Dt} + \frac{p}{\rho}\nabla \cdot \mathbf{u} = 0 \tag{8}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial r} \tag{9}$$

$$\nabla \cdot \mathbf{u} \equiv \frac{1}{r\cos\phi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial (v\cos\phi)}{\partial \phi}\right) + \frac{1}{r^2}\frac{\partial (r^2 w)}{\partial r} \tag{10}$$

3 Hierarchies of approximate equation sets

- **(Quasi-)hydrostatic**: Neglect Dw/Dt term. One of the thermodynamic equations then effectively becomes a diagnostic equation for w, so there are really only three component prognostic equations. The resulting equations no longer support internal acoustic waves, though they do support purely horizontally propagating external acoustic waves. Scale analysis shows that the hydrostatic approximation is valid on horizontal scales large compared with a typical depth scale for the atmosphere (about 10 km), or, equivalently, on timescales long compared with $1/N$ where N is the buoyancy frequency.

- **Shallow atmosphere**: Neglect the Coriolis terms proportional to $\cos\phi$ and certain related nonlinear terms (called the *traditional approximation*), and approximate r by the constant a equal to the mean radius of the Earth. These two approximations must be made together, otherwise the resulting equations fail to conserve energy and angular momentum.

- **Hydrostatic primitive equations**: Make both the hydrostatic and shallow atmosphere approximations.

- **Anelastic equations**: Essentially, neglect $\partial\rho/\partial t$. Then only two components of the momentum equation can be considered as independent prognostic equations. Moreover, the momentum equations combined with the density equation imply a diagnostic equation for p and hence ρ, so there are really only three component prognostic equations. There are various versions of the anelastic equations. They do not support acoustic waves. They are valid on short horizontal length scales.

- **Boussinesq**: Assume the fluid is incompressible $\nabla.\mathbf{u} = 0$, and neglect variations in density except where they appear in the buoyancy term. These equations do not support acoustic waves.

- **Quasigeostrophic**: Assume hydrostatic balance, Rossby number $Ro = U/(fL)$ is small, temperature is not far from some reference profile, and horizontal length scale is small compared with the Earth's radius. Usually the geometry is approximated as a β-plane. The governing equations reduce to a prognostic equation for the quasigeostrophic potential vorticity (plus another for boundary potential temperature), plus diagnostic equations relating the potential vorticity to winds and temperature. These equations do not support acoustic or gravity waves.

- **Planetary geostrophic**: Assume Rossby number $Ro = U/(fL)$ is small, and that the horizontal length scale is comparable to the Earth's radius. Spherical geometry may be retained. The governing equations reduce to a prognostic equation for the potential vorticity, plus diagnostic equations relating the potential vorticity to winds and temperature. These equations do not support acoustic or gravity waves.

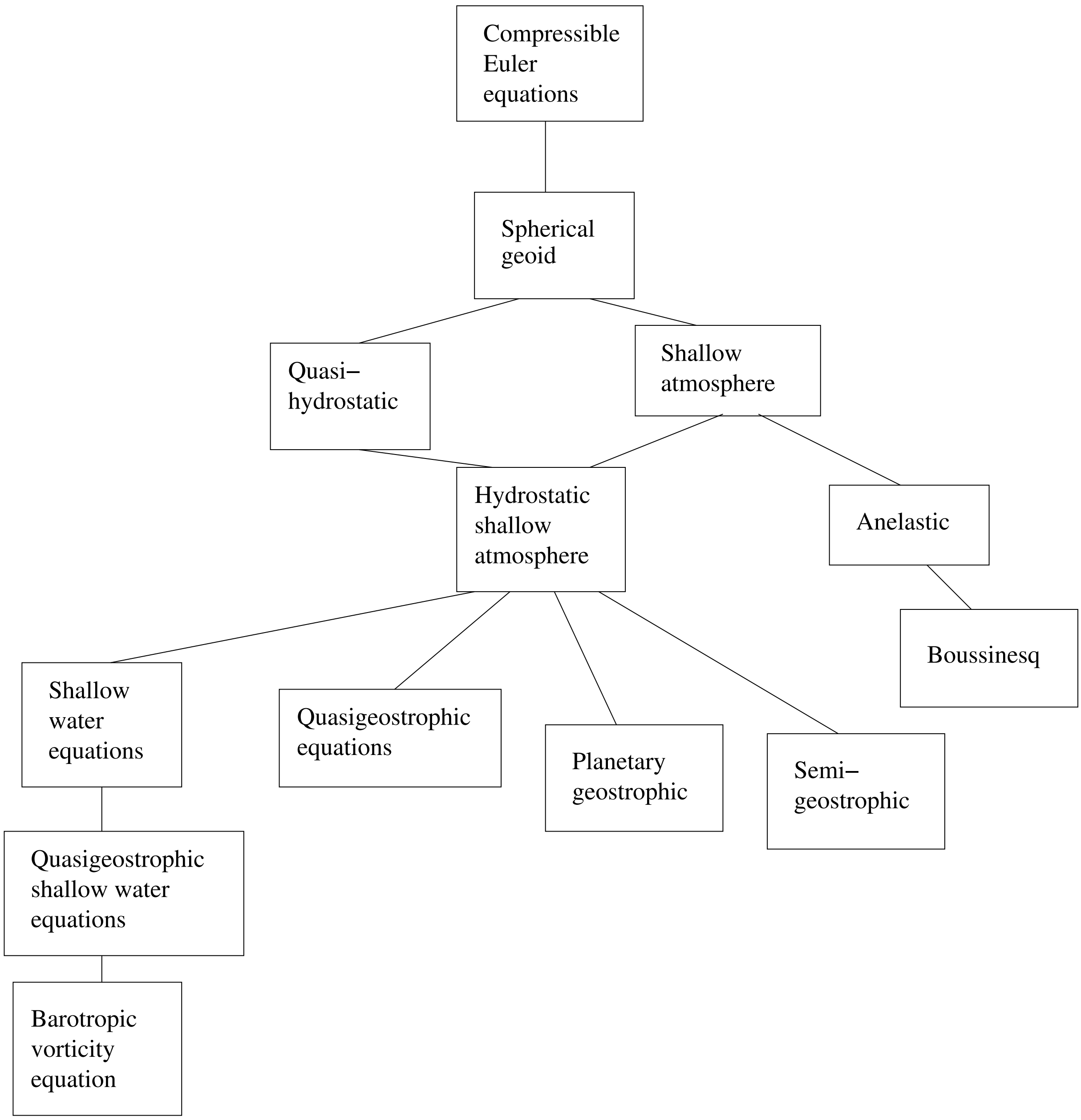

Figure 1: Part of the hierarchy of approximations to the full spherical-geometry compressible Euler equations.

- **Semi-geostrophic**: Assume Rossby number $Ro = U/(fL)$ is small, and horizontal length scale is very small compared with the Earth's radius so that the geometry can be approximated as an f-plane. These equations do not support acoustic or gravity modes.

- **Shallow water**: Assume an incompressible fluid in hydrostatic balance such that horizontal velocity is independent of z. The resulting equations are horizontally two dimensional. They do not support acoustic waves but do support horizontally propagating gravity and Rossby waves.

4 Hydrostatic balance

The following tables (based on similar tables in Holton's book) give typical scales of terms in the governing equations for mid-latitude synoptic scale flow.

Typical scales for the horizontal momentum equations

u-equation	$\frac{Du}{Dt}$	$\frac{uv\tan\phi}{r}$	$\frac{uw}{r}$	$-2\Omega v\sin\phi$	$2\Omega w\cos\phi$	$\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda}$
v-equation	$\frac{Dv}{Dt}$	$\frac{u^2\tan\phi}{r}$	$\frac{vw}{r}$	$2\Omega u\sin\phi$		$\frac{1}{\rho r}\frac{\partial p}{\partial\phi}$
Scales	U^2/L	U^2/a	WU/a	f_0U	f_0W	$\delta P/\rho L$
Values (ms^{-2})	10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-6}	10^{-3}

Typical scales for the vertical momentum equation

w-equation	$\frac{Dw}{Dt}$	$-\frac{u^2+v^2}{r}$	$-2\Omega u\cos\phi$	g	$\frac{1}{\rho}\frac{\partial p}{\partial r}$
Scales	UW/L	U^2/a	f_0U	g	$P_0/\rho H$
Values (ms^{-2})	10^{-7}	10^{-5}	10^{-3}	10	10

For mid-latitude synoptic scale flow, the dominant balance in the vertical momentum equation is clearly hydrostatic balance:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0. \tag{11}$$

We can justify neglecting the Dw/Dt term when

$$\frac{UW}{L} \ll \frac{\delta P}{\rho H} \tag{12}$$

where δP is typical horizontal variation in p. Assuming, from the horizontal momentum equation, that

$$\frac{\delta P}{\rho} \sim U^2 \text{ or } \frac{\delta P}{\rho} \sim f_0UL = U^2R_o^{-1}, \tag{13}$$

we require

$$\frac{WH}{UL} \ll 1 \text{ or } \frac{WH}{UL}R_o \ll 1. \tag{14}$$

Assuming further, from the mass continuity equation, that

$$\frac{W}{U} \sim \frac{H}{L} \text{ or } \frac{W}{U} \sim \frac{H}{L}R_o, \tag{15}$$

we see that hydrostatic balance will be a good approximation for

$$\frac{H^2}{L^2} \ll 1 \quad\text{or}\quad R_o^2\frac{H^2}{L^2} \ll 1 \tag{16}$$

where $R_o \equiv U/(f_0L)$ is the Rossby number.

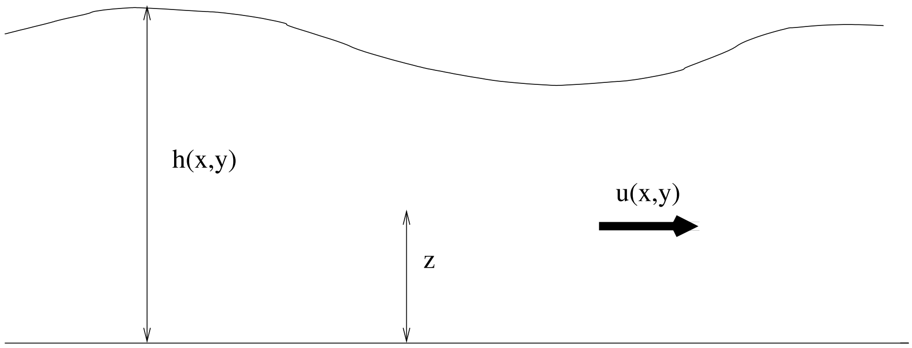

Figure 2: Schematic for derivation of shallow water equations.

5 The shallow water equations

Although not quantitatively accurate for most atmospheric or oceanic modelling, the shallow water equations embody many of the same physical ingredients as the full governing equations, and so are valuable for developing a conceptual understanding, as well as for testing numerical integration schemes.

5.1 Derivation

Consider a layer of incompressible fluid of constant density ρ in hydrostatic balance (see figure 2). Assume planar geometry and neglect $2\mathbf{\Omega}\cos\phi$ Coriolis terms. From hydrostatic balance, the pressure at any point in the fluid is given by

$$p(x, y, z) = \int_z^h \rho g \, dz = \rho g(h - z),$$

so the horizontal pressure gradient is

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\frac{\partial h}{\partial x}.$$

This is independent of z, so if u and v are initially independent of z then they will remain so.

Write $gh = \Phi$. Then the horizontal momentum equations become

$$\frac{Du}{Dt} - fv + \frac{\partial \Phi}{\partial x} = 0 \tag{17}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \Phi}{\partial y} = 0 \tag{18}$$

For later, note the vector form:

$$\frac{D\mathbf{v}}{Dt} + f\widehat{\mathbf{z}} \times \mathbf{v} + \nabla_{\mathrm{H}}\Phi = 0. \tag{19}$$

Take the vertical integral of the mass equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

to obtain

$$\int_0^h \frac{\partial u}{\partial x}\, dz + \int_0^h \frac{\partial v}{\partial y}\, dz + w(h) - w(0) = 0$$

$$h\frac{\partial u}{\partial x} + h\frac{\partial v}{\partial y} + \frac{Dh}{Dt} = 0.$$

Multiplying by g gives

$$\frac{D\Phi}{Dt} + \Phi\frac{\partial u}{\partial x} + \Phi\frac{\partial v}{\partial y} = 0. \tag{20}$$

or, rearranging,

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x}(u\Phi) + \frac{\partial}{\partial y}(v\Phi) = 0. \tag{21}$$

5.2 Vorticity and divergence equations

Take $\partial/\partial x$ of (18) minus $\partial/\partial y$ of (17) to obtain

$$\frac{D\zeta}{Dt} + \delta\zeta = 0 \tag{22}$$

or

$$\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}(u\zeta) + \frac{\partial}{\partial y}(v\zeta) = 0,$$

and take $\partial/\partial x$ of (17) plus $\partial/\partial y$ of (17) to obtain

$$\frac{\partial\delta}{\partial t} + \nabla_{\mathrm{H}} \cdot \left[\widehat{\mathbf{z}} \times \mathbf{v}\zeta + \nabla_{\mathrm{H}}\left(\Phi + \frac{u^2+v^2}{2}\right)\right] \tag{23}$$

where $\zeta = f + \xi$ is the absolute vorticity, $\xi = \partial v/\partial x - \partial u/\partial y$ is the relative vorticity, and $\delta = \partial u/\partial x + \partial v/\partial y$ is the divergence.

5.3 Potential vorticity equation

Combining the vorticity equation (22) with the mass equation (20) gives

$$\frac{D\Pi}{Dt} = 0 \tag{24}$$

where $\Pi = \zeta/\Phi$.

5.4 Circulation theorem

Integrating the vorticity equation within a closed material contour shows that

$$\frac{D\mathcal{C}}{Dt} = 0 \tag{25}$$

where

$$\mathcal{C} = \oint \mathbf{v}_{\mathrm{IF}} \cdot d\mathbf{l} = \int \zeta\, dA.$$

5.5 Energy equation

Take $u\Phi$ times (17) plus $v\Phi$ times (18) plus $\Phi + (u^2+v^2)/2$ times (21) to obtain

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(uE + \frac{u\Phi^2}{2}\right) + \frac{\partial}{\partial y}\left(vE + \frac{v\Phi^2}{2}\right) = 0 \tag{26}$$

where

$$E = \frac{\Phi^2}{2} + \Phi\left(\frac{u^2+v^2}{2}\right).$$

5.6 Normal modes: Poincaré waves and Rossby waves

Linearize the shallow water equations about a resting state with mean geopotential Φ_0 and take f to be constant. It is easiest to work with the vorticity and divergence equations:

$$\frac{\partial \xi}{\partial t} + f\delta = 0,$$

$$\frac{\partial \delta}{\partial t} - f\xi + \nabla_{\mathrm{H}}^2 \Phi = 0,$$

$$\frac{\partial \Phi}{\partial t} + \Phi_0 \delta = 0.$$

All coefficients in these equations are independent of space and time, so they will have solutions proportial to $e^{i(kx+ly-\omega t)}$. Seeking solutions of this form and eliminating ξ, δ, and Φ leads to the *dispersion relation*:

$$\omega\left(\omega^2 - f^2 - K^2\Phi_0\right) = 0$$

where $K^2 = k^2 + l^2$.

There are two kinds of solution, corresponding to the two kinds of root of the dispersion relation.

1. **Rossby waves**, corresponding to $\omega = 0$. These have $\delta \equiv 0$, and the wind field is in *geostrophic balance*:

$$v = \frac{1}{f}\frac{\partial \Phi}{\partial x}, \qquad u = -\frac{1}{f}\frac{\partial \Phi}{\partial y}.$$

   (Their frequency is zero because there is no β-effect; see later.)

2. **Inertio-gravity waves**, also called (in the shallow water case) **Poincaré waves**. These satisfy $\omega = \pm\left(f^2 + K^2\Phi_0\right)^{1/2}$. The two roots correspond to eastward and westward propagating waves. The potential vorticity perturbation is zero for these waves.

5.7 Phase velocity

For a wavelike disturbance (in one, two, or three dimensions) proportional to $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, individual wave crests move with the *phase velocity* $\mathbf{c}_{\mathrm{p}} = \mathbf{k}\omega/\left|\mathbf{k}\right|^2$.

5.8 Group velocity

Wave packets move at the group velocity

$$\mathbf{c}_{\mathrm{g}} = \nabla_{\mathbf{k}}\omega = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m}\right).$$

It is the velocity at which wave energy propagates.

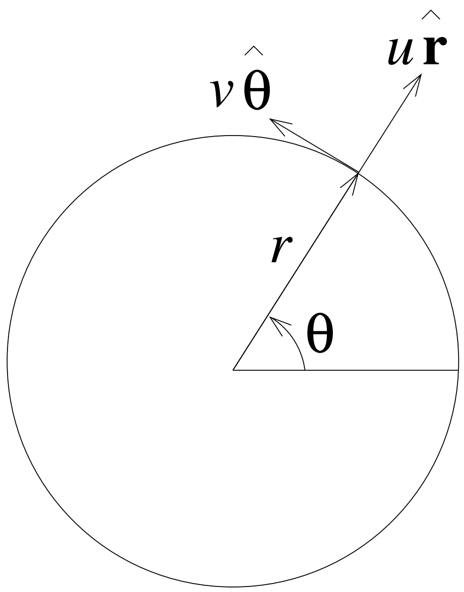

Figure 3: Schematic showing velocity components in polar coordinates.

5.9 Geostrophic balance

For small *Rossby number* $\mathrm{Ro} \equiv U/fL$, the dominant terms in the horizontal momentum equations are in geostrophic balance:

$$fv \approx fv_{\mathrm{g}} \equiv \frac{\partial \Phi}{\partial x},$$

$$fu \approx fu_{\mathrm{g}} \equiv -\frac{\partial \Phi}{\partial y}.$$

5.10 Gradient wind balance

If the flow is approximately steady but some of the nonlinear terms are non-negligible we might nevertheless have balanced flow.

Consider a circular vortex, with f constant, and work in polar coordinates (see figure 3). The vector horizontal velocity in polar coordinates is

$$\mathbf{v} = u\widehat{\mathbf{r}} + v\widehat{\boldsymbol{\theta}},$$

so now

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} &= \widehat{\mathbf{r}}\frac{Du}{Dt} + u\frac{D\widehat{\mathbf{r}}}{Dt} + \widehat{\boldsymbol{\theta}}\frac{Dv}{Dt} + v\frac{D\widehat{\boldsymbol{\theta}}}{Dt} \\ &= \widehat{\mathbf{r}}\frac{Du}{Dt} + \frac{uv}{r}\widehat{\boldsymbol{\theta}} + \widehat{\boldsymbol{\theta}}\frac{Dv}{Dt} - \frac{v^2}{r}\widehat{\mathbf{r}} \end{aligned}$$

Hence the momentum equation (19) becomes, in component form,

$$\frac{Du}{Dt} - \frac{v^2}{r} - fv + \frac{\partial \Phi}{\partial r} = 0, \tag{27}$$

$$\frac{Dv}{Dt} + \frac{uv}{r} + fu + \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = 0. \tag{28}$$

For a steady, circular vortex, $\partial/\partial\theta = 0$, $u = 0$,

$$\frac{v^2}{r} + fv = \frac{\partial \Phi}{\partial r}.$$

This is *gradient wind balance.*

In the limit of large Ro we get *cyclostrophic balance*:

$$\frac{v^2}{r} = \frac{\partial \Phi}{\partial r}.$$

5.11 Invertibility

If we know the distribution of potential vorticity, and we know that the flow is in balance, then we can deduce the mass and wind fields. This is the idea of *invertibility.*

For example, for linear shallow water flow

$$\frac{\xi_{\mathrm{b}}}{\Phi_0} - f\frac{\Phi_{\mathrm{b}}}{\phi_0^2} = \Pi$$

in geostrophic balance

$$f\xi_{\mathrm{b}} = \nabla_{\mathrm{H}}^2 \Phi_{\mathrm{b}}$$

Φ_{b}, and hence the balanced windcomponents, can be found by solving the elliptic equation

$$a^2 \nabla_{\mathrm{H}}^2 \Phi_{\mathrm{b}} - \Phi_{\mathrm{b}} = \frac{\Phi_0^2}{f}\Pi. \tag{29}$$

Here, $a = \sqrt{\Phi_0}/f$ is the *Rossby radius*, a natural length scale for geostrophically balanced flow.

5.12 Geostrophic adjustment

A flow that is disturbed away from balance can be considered to be made up of a superposition of a balanced (vortical or Rossby mode) component, and an unbalanced (inertio-gravity mode) component. Note we're neglecting the possibility of nonlinear interaction between the balanced and unbalanced components. The flow can *adjust* towards balance by radiating away the inertio-gravity wave component to leave just the balanced component. The final balanced flow can be found from the initial potential vorticity, using invertibility, since (linear) inertio-gravity modes have no potential vorticity perturbation. In general the inertio-gravity waves will carry away energy, so that the final balanced flow will have less energy (locally) than the initial flow.

5.13 Quasigeostrophic shallow water flow

We make the following three assumptions:

1. the motion is nearly geostrophic: Ro $\ll 1$;

2. fractional changes in the Coriolis parameter $f = f_0 + \beta y$ are small on the horizontal length scale L of the flow: $\beta L / f_0 \ll 1$;
3. fractional changes in the total depth are small: $|(\Phi - \Phi_0)/\Phi_0| \ll 1$, where Φ_0 is a constant mean depth.

The evolution of the flow can then be expressed in terms of the *quasigeostrophic potential vorticity equation*

$$\frac{D_g q}{Dt} = 0 \tag{30}$$

where

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

and

$$\begin{aligned} q &= \zeta_g - \frac{f_0^2}{\Phi_0}\psi \\ &= f_0 + \beta y + \nabla_{\mathrm{H}}^2 \psi - \frac{1}{a^2}\psi \end{aligned} \tag{31}$$

is the quasigeostrophic potential vorticity, with a again the Rossby radius. Here ψ is the geostrophic stream function; it is related to the wind and mass fields by

$$u_g = -\frac{\partial \psi}{\partial y},$$

$$v_g = \frac{\partial \psi}{\partial x},$$

and

$$\Phi_1 = \Phi - \Phi_0 = f_0 \psi.$$

The dynamics is determined by the advection (30) and inversion (31) of quasigeostrophic potential vorticity. The quasigeostrophic equations do not support gravity waves: they have been filtered out by neglecting the $D\mathbf{v}/Dt$ term at leading order.

5.14 The "omega" equation

The "omega" equation provides a robust way to determine the divergence δ, by asking what δ is needed to maintain geostrophic balance as the flow evolves. Taking the time derivative of

$$f_0 \xi_g = \nabla_{\mathrm{H}}^2 \Phi_1$$

and substituting from the vorticity and mass equations leads to the "omega" equation.

$$\left(a^2 \nabla_{\mathrm{H}}^2 - 1\right)\delta = \frac{1}{f_0}\left(\mathbf{v}_g \cdot \nabla_{\mathrm{H}} \xi_g + \beta v_g\right) - \frac{1}{f_0^2}\nabla_{\mathrm{H}}^2\left(\mathbf{v}_g \cdot \nabla_{\mathrm{H}} \Phi_1\right). \tag{32}$$

The operator on the left hand side is an elliptic operator, implying that the response to the right-hand-side forcing will be nonlocal. The first term on the right hand side comes from vorticity advection; the second term on the right hand side comes from mass advection.

The analogous problem in the three-dimensional pressure-coordinate case is to determine the vertical velocity ω; hence the name "omega" equation.

5.15 QG Rossby waves

Consider small perturbations to a state of rest, so that we can linearize the QGPV equation

$$\frac{\partial q'}{\partial t} + \beta v' = 0.$$

Seeking wavelike solutions

$$\psi' = \widehat{\psi} e^{i(kx+ly-\omega t)},$$

$$q' = -\left(K^2 + \frac{1}{a^2}\right)\psi',$$

leads to the Rossby wave dispersion relation

$$\omega = -\frac{k\beta}{K^2 + 1/a^2}. \tag{33}$$

Rossby waves propagate westwards (relative to any background flow).

6 Gravity waves

6.1 The Boussinesq approximation

The Boussinesq approximation involves assuming the fluid to be incompressible ($\nabla \cdot \mathbf{u} = 0$), and neglecting variations in density except where they appear in a buoyancy term, i.e. multiplied by g. Here we will also approximate the geometry as Cartesian.

$$\frac{Du}{Dt} - fv + Fw = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{34}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \tag{35}$$

$$\frac{Dw}{Dt} - Fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho_0}, \tag{36}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{37}$$

$$\frac{D\rho'}{Dt} + w\frac{\partial \rho_0}{\partial z} = 0. \tag{38}$$

Here, $f = 2\Omega \sin\phi$ and $F = 2\Omega\cos\phi$, and we approximate ϕ as a fixed latitude.

It is convenient to work in terms of the *buoyancy* $b = -g\rho'/\rho_0$, so that (38) becomes

$$\frac{Db}{Dt} + wN^2 = 0, \tag{39}$$

where

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$$

is a measure of the stratification.

6.2 Internal gravity waves

Linearize the Boussinesq equations (34)-(39) about a state of rest, and neglect the Coriolis terms. [Ex: leave the Coriolis terms in!]

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{40}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \tag{41}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - b, \tag{42}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{43}$$

$$\frac{\partial b}{\partial t} + wN^2 = 0. \tag{44}$$

Seek wavelike solutions proportial to $e^{i(kx+ly+mz-\omega t)}$, and eliminate u, v, w, b and p to obtain the dispersion relation

$$\omega^2 = \frac{K^2N^2}{K^2+m^2}, \tag{45}$$

where $K^2 = k^2 + l^2$ is the total horizontal wavenumber squared.

It can be shown that

1. these waves are *transverse* i.e., $\mathbf{k}\cdot\mathbf{u} = 0$;
2. the buoyancy b is in quadrature with the vertical velocity w;
3. the vertical components of $\mathbf{c}_\mathrm{p}$ and $\mathbf{c}_\mathrm{g}$ have opposite sign—upward energy propagation requires downward phase propagation;
4. $\mathbf{c}_\mathrm{p}\cdot\mathbf{c}_\mathrm{g} = 0$—energy propagation is orthogonal to phase propagation.

6.3 Mountain waves

So far we have considered *free waves*: given a disturbance with wave vector (k,l,m), how does it evolve? We now consider *forced waves*: given an imposed disturbance pattern with horizontal wave vector (k,l) and frequency ω at a bottom boundary, how does the disturbance propagate away from the boundary?

Retain the Boussinesq equations, but now linearize about a mean flow $(U,0,0)$ with U a constant. The dispersion relation becomes

$$\tilde{\omega}^2 \equiv (\omega - kU)^2 = \frac{K^2N^2}{K^2+m^2}. \tag{46}$$

$\tilde{\omega}$ is called the *intrinsic frequency.*

Suppose we have a solid bottom boundary at height $z = h_\mathrm{M}(x)$. The no-normal-flow boundary condition becomes, assuming that $h_\mathrm{M}(x)$ is small so that we can linearize,

$$w(z=0) = U\frac{dh_\mathrm{M}}{dx}. \tag{47}$$

Now suppose the bottom boundary forcing is wavelike $h_\mathrm{M} = \Re\left\{\hat{h}_\mathrm{M}e^{ikx}\right\}$, with frequency $\omega = 0$. The flow will then have a wavelike response, with the same k, $l = 0$, $\omega = 0$, and m determined by the need to satify the dispersion relation:

$$m^2 = \frac{N^2}{U^2} - k^2.$$

Two types of solution are possible.

1. **Propagating waves**. Provided $k^2U^2 < N^2$, m^2 will be positive and m will be real; the solution will be wavelike in the vertical direction. Physically, we must choose the root that gives upward group velocity and energy propagation.
2. **Trapped waves**. If, on the other hand, $k^2U^2 > N^2$, m^2 will be negative and m purely imaginary. We must choose the root with $\Im\{m\} > 0$ so that the disturbance amplitude decays away from the boundary. Wave energy is trapped near the boundary.

6.4 Mountain wave drag

Intuitively we might expect the presence of hills to slow the flow, i.e. to exert a drag. This will happen if a pressure gradient exists across the hills.

The eastward force exerted by the ground on the atmosphere per unit length is

$$\begin{aligned}
\tau_0 &= -\overline{p' \frac{dh_{\mathrm{M}}}{dx}} \\
&= -\left(\frac{k}{2\pi}\right) \int_{\text{one wavelength}} p' \frac{dh_{\mathrm{M}}}{dx}\, dx, \\
&= \frac{1}{2} \Re \left\{ ik\hat{p}\hat{h}_{\mathrm{M}}^* \right\}
\end{aligned} \tag{48}$$

where the integral is taken at $z = 0$, and asterisk means complex conjugate.

From the linearized bottom boundary condition and the continuity equation we can relate $\hat{p}$ to $\hat{h}_{\mathrm{M}}$:

$$\hat{p} = \rho_0 imU^2 \hat{h}_{\mathrm{M}}.$$

Then subtituting in (48) gives an expression for τ_0 in terms of the mountain height and the mean wind:

$$\tau_0 = -\frac{1}{2}\rho_0 U^2 \Re \left\{ mk \left| \hat{h}_{\mathrm{M}} \right|^2 \right\}.$$

For upward propagating waves with $U > 0$ m is real and positive, so $\tau_0 < 0$: the ground exerts a drag on the flow.

It can be shown that the upward flux of eastward momentum

$$\begin{aligned}
\tau &= \rho_0 \overline{uw} \\
&= \rho_0 \left(\frac{k}{2\pi}\right) \int_{\text{one wavelength}} uw\, dx, \\
&= \frac{1}{2}\rho_0 \Re \left\{ \hat{u}\hat{w}^* \right\}
\end{aligned}$$

is equal to τ_0 and is independent of z. Thus, for the linear, conservative flow described by these equations, the wave-induced upward flux of eastward momentum is conserved. At some altitude, however, dissipative effects or nonlinear effects will become important, and there the waves exert a drag on the mean flow given by

$$-\frac{1}{\rho_0}\frac{\partial \tau}{\partial z}.$$

7 The hydrostatic primitive equations

7.1 The shallow atmosphere approximation

Motivated by the fact that $H \ll a$ (here a is the Earth's radius) we make the following approximations:

1. drop the $2\Omega \cos\phi$ Coriolis terms (the *traditional approximation*);

2. drop the "spherical metric terms";
3. replace r by the constant a (the Earth's mean radius) and $\partial/\partial r$ by $\partial/\partial z$.

This appears to be a good approximation for the earth's atmosphere, except possibly for deep, diabatically forced motions, or when N^2 is very small.

Note we must make all three approximations together to retain conservation laws for energy, angular momentum, and potential vorticity.

7.2 The quasi-hydrostatic approximation

We make the quasi-hydrostatic approximation by neglecting the $\partial w/\partial t$ term in the vertical momentum equation. One of the thermodynamic equations then becomes, in effect, a diagnostic equation for w. This approximation filters out internal acoustic modes. We retain conservation laws for energy, angular momentum and potential vorticity, but the conserved energy no longer includes the $w^2/2$ contribution. Scaling arguments suggest that the quasi-hydrostatic approximation should be valid for $H^2/L^2 \ll 1$, or $\omega^2 \ll N^2$.

7.3 The hydrostatic primitive equations

The shallow atmosphere and quasi-hydrostatic approximations are independent of each other. If we make them both together we obtain the *hydrostatic primitive equations*, which are widely used for climate modelling and numerical weather prediction.

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} - 2\Omega v\sin\phi + \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} = 0 \tag{49}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + 2\Omega u\sin\phi + \frac{1}{\rho a}\frac{\partial p}{\partial\phi} = 0 \tag{50}$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{51}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{52}$$

$$c_v\frac{DT}{Dt} + \frac{p}{\rho}\nabla\cdot\mathbf{u} = 0 \tag{53}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z} \tag{54}$$

$$\nabla\cdot\mathbf{u} \equiv \frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right) + \frac{\partial w}{\partial z} \tag{55}$$

7.4 Hydrostatic primitive equations in a pressure coordinate

Under the hydrostatic approximation, the pressure difference across a layer of fluid is proportional to the mass per unit area of fluid in the layer. Using p as a vertical coordinate helps bring out conservation properties, especially in numerical models.

The z-coordinate hydrostatic primitive equations appear to have four prognostic variables. However, the hydrostatic equation implies a relation between two thermodynamic variables: one of the thermodynamic prognostic equations must be re-interpreted as a diagnostic equation for w. This issue is much clearer in a pressure coordinate, where the mass continuity equation is a purely diagnostic equation.

We can transform to a pressure coordinate using the transformation rules:

$$\left.\frac{\partial\psi}{\partial s}\right|_z = \left.\frac{\partial\psi}{\partial s}\right|_p - \left.\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial s}\right|_p$$

and

$$\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial p}\frac{\partial p}{\partial z},$$

where s stands for any of x, y, or t, and ψ may be any field of interest.

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} - 2\Omega v\sin\phi + \frac{1}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} = 0 \tag{56}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\Phi}{a} + 2\Omega u\sin\phi + \frac{1}{a}\frac{\partial\phi}{\partial\phi} = 0 \tag{57}$$

$$\frac{\partial\Phi}{\partial p} = -\frac{RT}{p} \tag{58}$$

$$\frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right) + \frac{\partial\omega}{\partial p} = 0 \tag{59}$$

$$\frac{DT}{Dt} + \frac{\kappa\omega T}{p} = 0 \tag{60}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial p}, \tag{61}$$

$$\omega \equiv \frac{Dp}{Dt}, \tag{62}$$

and horizontal derivatives are now understood to be taken at constant p rather than constant z.

The hydrostatic equations in a pressure coordinate, or in a terrain-following variant of the pressure coordinate, are often used in numerical models.

7.5 Log-pressure coordinate

Closely related is the use of a log-pressure coordinate

$$\tilde{z} = -H_\rho \ln(p/p_{\mathrm{ref}})$$

where $H_\rho = RT_{\mathrm{ref}}/g$ is a reference density scale height, with T_{ref} a constant mean temperature and p_{ref} a constant mean surface pressure.

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} - 2\Omega v\sin\phi + \frac{1}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda} = 0 \tag{63}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\Phi}{a} + 2\Omega u\sin\phi + \frac{1}{a}\frac{\partial\phi}{\partial\phi} = 0 \tag{64}$$

$$\frac{\partial\Phi}{\partial\tilde{z}} = \frac{RT}{H_\rho} = g\frac{T}{T_{\mathrm{ref}}} \tag{65}$$

$$\frac{1}{a\cos\phi}\left(\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right) + \frac{1}{p}\frac{\partial}{\partial\tilde{z}}(p\tilde{w}) = 0 \tag{66}$$

$$\frac{D\theta}{Dt} = 0 \tag{67}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + \tilde{w}\frac{\partial}{\partial\tilde{z}} \tag{68}$$

and

$$\tilde{w} \equiv \frac{D\tilde{z}}{Dt}. \tag{69}$$

8 Thermal wind relation

The geostrophic relation

$$fv_g = \frac{\partial\Phi}{\partial x}; \qquad fu_g = -\frac{\partial\Phi}{\partial y},$$

may be combined with the hydrostatic relation (65) to obtain the thermal wind relation

$$f\frac{\partial v_g}{\partial\tilde{z}} = \frac{R}{H_\rho}\frac{\partial T}{\partial x}; \qquad f\frac{\partial u_g}{\partial\tilde{z}} = -\frac{R}{H_\rho}\frac{\partial T}{\partial y}, \tag{70}$$

or, in vector form,

$$f\hat{\mathbf{z}} \times \frac{\partial\mathbf{v}_g}{\partial\tilde{z}} = \frac{R}{H_\rho}\nabla T = \frac{g}{T_{\mathrm{ref}}}\nabla T. \tag{71}$$

9 Quasigeostrophic theory

Starting from the hydrostatic primitive equations in a log-pressure coordinate, (for which we require $H^2 \ll L^2$), we make the following further scaling assumptions:

1. small Ro, so that $\mathrm{v} \approx \mathrm{v}_g + O(Ro)$;
2. mid-latitude β-plane geometry, and $\beta L/f_0 = O(Ro) \ll 1$;
3. $\theta \approx \theta_0(\tilde{z}) + \theta'(x, y, \tilde{z}, t)$, with $\theta'/\theta_0 = O(Ro) \ll 1$.

(Compare what we assumed in deriving the quasigeostrophic shallow water equations.)

The leading terms in the momentum equations are purely diagnostic:

$$f_0 v_g = \frac{\partial \Phi}{\partial x}; \qquad f_0 u_g = -\frac{\partial \Phi}{\partial y}.$$

(Recall the horizontal derivatives are at constant $\tilde{z}$.)

Since v_g is non-divergent we can introduce a stream function $\psi = \Phi'/f_0$, where $\Phi = \Phi_0(\tilde{z}) + \Phi'(x, y, \tilde{z}, t)$, Φ_0 is in hydrostatic balance with θ_0, and Φ' is the horizontally varying part of Φ. Then

$$v_g = \frac{\partial \psi}{\partial x}; \qquad u_g = -\frac{\partial \psi}{\partial y}.$$

To obtain prognostic equations we need to go to higher order. The next order terms in the mass continuity equation are

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{p}\frac{\partial}{\partial z}(p\tilde{w}) = 0.$$

Thus, $\tilde{w}$ is smaller, by order Ro, than the obvious scaling UH/L. This means we can neglect vertical advection terms

$$\begin{aligned} \frac{D}{Dt} &\equiv \tfrac{\partial}{\partial t} + u\tfrac{\partial}{\partial x} + v\tfrac{\partial}{\partial y} + \tilde{w}\tfrac{\partial}{\partial \tilde{z}} \\ &\approx \tfrac{\partial}{\partial t} + u_g\tfrac{\partial}{\partial x} + v_g\tfrac{\partial}{\partial y} \equiv \frac{D_g}{Dt} \end{aligned}$$

except in the θ equation.

The next order terms in the momentum equations give

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0, \tag{72}$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0, \tag{73}$$

while the leading non-trivial terms in the θ equation are

$$\frac{D_g \theta'}{Dt} + \tilde{w}\frac{\partial \theta_0}{\partial \tilde{z}} = 0$$

or, equivalently,

$$\frac{D_g}{Dt}\left(\frac{\theta'}{\theta_{\mathrm{ref}}}\right) + \tilde{w}\frac{N_{\mathrm{ref}}^2}{g} = 0, \tag{74}$$

where

$$N_{\mathrm{ref}}^2 = \frac{g}{\theta_{\mathrm{ref}}}\frac{\partial \theta_0}{\partial \tilde{z}}; \qquad \theta_{\mathrm{ref}}(\tilde{z}) = T_{\mathrm{ref}}\left(\frac{p}{p_{\mathrm{ref}}}\right)^{\kappa}.$$

9.1 Quasigeostrophic vorticity equation

Take $\partial/\partial x$ (73) minus $\partial/\partial y$ (72), and use the fact that $\partial u_g/\partial x + \partial v_g/\partial y = 0$ to obtain the quasigeostrophic vorticity equation

$$\frac{D_g \xi_g}{Dt} + \beta v_g = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right),$$

where $\xi_g = \partial v_g/\partial x - \partial u_g/\partial y$ is the geostrophic approximation to the vertical component of relative vorticity. This may be re-written as

$$\frac{D_g \zeta_g}{Dt} = -f_0 \delta,$$

where $\zeta_g = f_0 + \beta y + \xi_g$ is the geostrophic approximation to the vertical component of absolute vorticity, and $\delta = \partial u_a/\partial x + \partial v_a/\partial y$ is the horizontal divergence. Then, substituting from the mass continuity equation, we have

$$\frac{D_g \zeta_g}{Dt} = \frac{f_0}{p}\frac{\partial}{\partial \tilde{z}}(p\tilde{w}). \tag{75}$$

The term on the right hand side is the quasigeostrophic vortex stretching term.

9.2 Quasigeostrophic potential vorticity equation

Use (74) to eliminate $\tilde{w}$ from the (75) and use the hydrostatic relation

$$\frac{\theta'}{\theta_{\mathrm{ref}}} = \frac{T'}{T_{\mathrm{ref}}} = \frac{1}{g}\frac{\partial \Phi'}{\partial \tilde{z}}$$

to eliminate θ':

$$\frac{D_g \zeta_g}{Dt} = -\frac{f_0}{p}\frac{\partial}{\partial \tilde{z}}\left\{\frac{p}{N_{\mathrm{ref}}^2}\frac{D_g}{Dt}\left(\frac{\partial \Phi'}{\partial \tilde{z}}\right)\right\}.$$

Finally use the fact that

$$\frac{\partial \mathbf{v}_g}{\partial \tilde{z}} \cdot \nabla_{\tilde{z}}\left(\frac{p}{N_{\mathrm{ref}}^2}\frac{\partial \Phi'}{\partial \tilde{z}}\right) = 0$$

to re-write the right hand side as

$$\frac{D_g \zeta_g}{Dt} = -\frac{D_g}{Dt}\left\{\frac{f_0}{p}\frac{\partial}{\partial \tilde{z}}\left(\frac{p}{N_{\mathrm{ref}}^2}\frac{\partial \Phi'}{\partial \tilde{z}}\right)\right\}.$$

In other words

$$\frac{D_g q}{Dt} = 0 \tag{76}$$

where

$$q = \zeta_g + \frac{f_0}{p}\frac{\partial}{\partial \tilde{z}}\left(\frac{p}{N_{\mathrm{ref}}^2}\frac{\partial \Phi'}{\partial \tilde{z}}\right)$$
$$= f_0 + \beta y + \nabla_{\tilde{z}}^2 \psi + \frac{1}{p}\frac{\partial}{\partial \tilde{z}}\left(\frac{p f_0^2}{N_{\mathrm{ref}}^2}\frac{\partial \psi}{\partial \tilde{z}}\right)$$

or, defining $\rho_{\mathrm{ref}} = p/RT_{\mathrm{ref}}$,

$$= f_0 + \beta y + \nabla_{\tilde{z}}^2 \psi + \frac{1}{\rho_{\mathrm{ref}}}\frac{\partial}{\partial \tilde{z}}\left(\rho_{\mathrm{ref}}\frac{f_0^2}{N_{\mathrm{ref}}^2}\frac{\partial \psi}{\partial \tilde{z}}\right). \tag{77}$$

The quantity q is the quasigeostrophic potential vorticity. Note that it is conserved following the geostrophic flow.

9.3 Invertibility

If we know the three-dimensional distribution of q and the profiles of $\rho_{\mathrm{ref}}(\tilde{z})$ and $N_{\mathrm{ref}}^2(\tilde{z})$ then, with suitable boundary conditions, we can solve the elliptic problem (77) for ψ, and hence determine

$$u_g = -\frac{\partial \psi}{\partial x}; \quad v_g = \frac{\partial \psi}{\partial y}; \quad \frac{T'}{T_{\mathrm{ref}}} = \frac{f_0}{g}\frac{\partial \psi}{\partial \tilde{z}}.$$

Many GFD problems can be understood qualitatively (and even quantitatively) in terms of advection and inversion of QGPV.

9.4 Omega equation

How can we determine the vertical velocity robustly from quantities that are relatively straightforward to observe?

We can ask what vertical velocity is required to ensure that the flow remains close to hydrostatic and geostrophic balance as it evolves (compare what we did in the shallow water case).

From the geostrophic vorticity equation

$$\frac{\partial}{\partial t}\left(\nabla_{\tilde{z}}^2 \psi\right) + \mathbf{v}_g \cdot \nabla_{\tilde{z}} \xi_g + \beta v_g = \frac{f_0}{p}\frac{\partial}{\partial \tilde{z}}\left(p\tilde{w}\right)$$

or, taking the vertical derivative,

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial \tilde{z}}\left(\nabla_{\tilde{z}}^2 \psi\right)\right] + \frac{\partial}{\partial \tilde{z}}(\mathrm{VA}) = f_0 \frac{\partial}{\partial \tilde{z}}\left(\frac{1}{p}\frac{\partial}{\partial \tilde{z}}\left(p\tilde{w}\right)\right),$$

where $\mathrm{VA} = \mathbf{v}_g \cdot \nabla_{\tilde{z}} \xi_g + \beta v_g$ is the vorticity advection term. At the same time, from the thermodynamic equation,

$$\frac{\partial}{\partial t}\left(\frac{\theta'}{\theta_{\mathrm{ref}}}\right) + \mathbf{v}_g \cdot \nabla_{\tilde{z}}\left(\frac{\theta'}{\theta_{\mathrm{ref}}}\right) + \frac{N_{\mathrm{ref}}^2}{g}\tilde{w} = 0,$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial \tilde{z}}\right) + \frac{g}{f_0}(\mathrm{TA}) + \frac{N_{\mathrm{ref}}^2}{f_0}\tilde{w} = 0,$$

where $\mathrm{TA} = \mathbf{v}_g \cdot \nabla_{\tilde{z}}\left(\frac{\theta'}{\theta_{\mathrm{ref}}}\right)$ is the thermal advection term. Taking the horizontal Laplacian of this last equation and eliminating the time derivative term leads to the "Omega equation"

$$\nabla_{\tilde{z}}^2\tilde{w} + \frac{f_0^2}{N_{\mathrm{ref}}^2}\frac{\partial}{\partial \tilde{z}}\left(\frac{1}{p}\frac{\partial}{\partial \tilde{z}}(p\tilde{w})\right) = \frac{g}{N_{\mathrm{ref}}^2}\nabla_{\tilde{z}}^2(\mathrm{TA}) - \frac{f_0}{N_{\mathrm{ref}}^2}\frac{\partial}{\partial \tilde{z}}(\mathrm{VA}).$$

This is an elliptic equation for $\tilde{w}$ in terms of the vorticity advection and thermal advection. Suitbale boundary conditions will be required to solve this equation. The response in $\tilde{w}$ will not be localized exactly where the forcing is. Note the elliptic operator is closely related to that relating q and ψ.

10 Vertical propagation of planetary waves

The summer stratosphere has very symmetrical, undisturbed westard flow, but the winter stratosphere's mean eastward flow is strongly disturbed by planetary scale waves (zonal wavenumbers 1 and 2).

Linearize the QGPV equation about a constant zonal flow $(U, 0)$:

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + \beta v' = 0$$

and seek stationary $\omega = 0$ wavelike solutions

$$\psi' = \Re\left\{\Psi(\tilde{z})e^{i(kx+ly)}\right\}$$

etc. Substituting solutions of this form, the linearized PV equation becomes

$$\frac{1}{\rho_{\mathrm{ref}}}\frac{\partial}{\partial \tilde{z}}\left(\rho_{\mathrm{ref}}\frac{\partial \Psi}{\partial \tilde{z}}\right) + \frac{N_{\mathrm{ref}}^2}{f_0^2}\left(\frac{\beta}{U} - (k^2 + l^2)\right)\Psi = 0,$$

where we have assumed N_{ref}^2 to be constant for simplicity.

Noting that $\rho_{\mathrm{ref}} \propto e^{-\tilde{z}/H_\rho}$, we can find solutions of the form

$$\Psi = \Psi_0 e^{(im+1/2H_\rho)\tilde{z}}$$

where m must satisfy

$$m^2 = \frac{N_{\mathrm{ref}}^2}{f_0^2}\left(\frac{\beta}{U} - (k^2 + l^2)\right) - \frac{1}{4H_\rho^2}.$$

Vertically propagating solutions, with real m, can exist provided the RHS is greater than zero. This will be true for U in the range

$$0 < U < \frac{\beta}{k^2 + l^2 + \frac{f_0^2}{N_{\mathrm{ref}}^2}\frac{1}{4H_\rho^2}}.$$

It may be verified that the upward propagating solution has $m > 0$. Thus, stationary planetary waves can propagate upward into the stratosphere only for $U > 0$ and provided U is not too strong. This is the *Charney-Drazin theorem.*

If the RHS is less than zero then m will be purely imaginary; waves are vertically trapped.

For any given $U > 0$ waves can propagate only for

$$k^2 + l^2 < \frac{\beta}{U} - \frac{f_0^2}{N_{\text{ref}}^2}\frac{1}{4H_{\rho 2}^2},$$

that is, only the waves of largest horizontal scale.

Finally, note that, because of the exponential growth with height, nonlinear effects must become important at some height, and then the linear analysis will no longer be valid.

10.1 QG waves on a boundary temperature gradient

Neglect β and vertical variations in ρ_{ref} so that $H_\rho \to \infty$ (a kind of Boussinesq approximation). Then the quasigeostrophic PV is given by

$$q = f_0 + \beta y + \nabla_{\tilde{z}}^2 \psi + \frac{f_0^2}{N_{\text{ref}}^2}\frac{\partial^2 \psi}{\partial \tilde{z}^2}. \tag{78}$$

Consider a basic state with a constant vertical shear $\partial U/\partial \tilde{z}$ in balance with a constant northward temperature gradient

$$\frac{\partial U}{\partial \tilde{z}} = -\frac{g}{f_0}\frac{1}{\theta_{\text{ref}}}\frac{\partial \Theta}{\partial y}.$$

Look for solutions in which the interior potential vorticity perturbation vanishes, but there may be non-zero potential temperature perturbations at the bottom boundary at $\tilde{z} = 0$. These are simply advected by the surface geostrophic wind, since $\tilde{w} = 0$ there:

$$\frac{D_g}{Dt}\left(\frac{\theta'}{\theta_{\text{ref}}}\right) = 0.$$

(Compare (74).)

Linearize the θ equation at the bottom boundary about the basic flow, and seek wavelike solutions $\propto e^{i(kx+ly+m\tilde{z}-\omega t)}$. The condition of zero interior PV perturbation implies

$$k^2 + l^2 + \frac{f_0^2}{N_{\text{ref}}^2} m^2 = 0,$$

so, for real k and l, m must be purely imaginary—solutions will be trapped in the vertical. Physically, we need $\Im\{m\} > 0$ so that the solution decays away from the bottom boundary. The linearized θ equation then gives

$$(\omega - kU(0))\, m = ik\frac{\partial U}{\partial \tilde{z}}.$$

These waves propagate at zonal phase speed

$$\begin{aligned} c_p &= \frac{\omega}{k} = U(0) + \frac{i}{m}\frac{\partial U}{\partial \tilde{z}} \\ &= U(0) + H_R \frac{\partial U}{\partial \tilde{z}} \\ &= U(H_R), \end{aligned}$$

where

$$H_R = \frac{i}{m} = \frac{f_0}{N_{\mathrm{ref}}(k^2 + l^2)^{1/2}}$$

is the *Rossby height*.

Thus, a boundary temperature gradient can support boundary-trapped Rossby waves. The temperature gradient plays a role somewhat analogous to β. It can be shown that the circulation is cyclonic above a warm surface temperature anomaly.

At a bottom boundary, $f_0 \partial\Theta/\partial y > 0$ implies westward wave propagation (relative to the surface wind), while $f_0 \partial\Theta/\partial y < 0$ implies eastward wave propagation (relative to the surface wind). At a top boundary the direction of propagation would be reversed.

10.2 Baroclinic instability: The Eady model

Again neglect β and vertical variabtions of ρ_{ref}, so that again the QGPV is given by (78). Again consider a constant background zonal shear and northward temperature gradient, but now bounded by lower and upper boundaries at $\tilde{z} = \pm H$.

Again, seek solutions with zero potential vorticity perturbation in the interior. The general solution for ψ is most conveniently written

$$\psi = \Re\left\{\hat{\psi} e^{i(kx+ly)}\left[A \sinh\left(\frac{\tilde{z}}{H_R}\right) + B \cosh\left(\frac{\tilde{z}}{H_R}\right)\right]\right\},$$

where $\hat{\psi}$ is a constant.

At the bottom boundary the θ equation becomes

$$\begin{aligned} -i\left[\omega - kU(-H)\right]\left[\frac{A}{H_R}\cosh\left(\frac{-H}{H_R}\right) + \frac{B}{H_R}\sinh\left(\frac{-H}{H_R}\right) +\right] & \\ -ik\frac{\partial U}{\partial \tilde{z}}\left[A \sinh\left(\frac{-H}{H_R}\right) + B\cosh\left(\frac{-H}{H_R}\right) +\right] &= 0, \end{aligned}$$

while at the top boundary it becomes

$$\begin{aligned} -i\left[\omega - kU(H)\right]\left[\frac{A}{H_R}\cosh\left(\frac{H}{H_R}\right) + \frac{B}{H_R}\sinh\left(\frac{H}{H_R}\right) +\right] & \\ -ik\frac{\partial U}{\partial \tilde{z}}\left[A \sinh\left(\frac{H}{H_R}\right) + B\cosh\left(\frac{H}{H_R}\right) +\right] &= 0. \end{aligned}$$

Using the fact that

$$U(\pm H) = \pm H \frac{\partial U}{\partial \tilde{z}},$$

and eliminating A and B leads to the dispersion relation

$$\omega^2 = k^2 \left|\frac{\partial U}{\partial \tilde{z}}\right|^2 \left[H \coth\left(\frac{H}{H_R}\right) - H_R\right] \left[H \tanh\left(\frac{H}{H_R}\right) - H_R\right].$$

For $H \gg H_R$ (i.e. k or l large—the short wave limit), $\coth(H/H_R) \approx 1$, $\tanh(H/H_R) \approx 1$,

$$\omega \approx \pm k \left|\frac{\partial U}{\partial \tilde{z}}\right| (H - H_R),$$

and

$$c_p \approx \pm \left|\frac{\partial U}{\partial \tilde{z}}\right| (H - H_R).$$

These are just the boundary waves we met in section 10.1.

For $H \ll H_R$ (the long wave limit)

$$\omega^2 \approx -\frac{1}{3} k^2 H^2 \left|\frac{\partial U}{\partial \tilde{z}}\right|^2 < 0;$$

ω is purely imaginary—we have solutions that grow or decay exponentially with time. The growing solutions are *unstable*. By considering the signs of the factors in the dispersion relation we can see that unstable solutions will exist for

$$H \tanh\left(\frac{H}{H_R}\right) < H_R,$$

i.e.,

$$\frac{H}{H_R} < 1.2.$$

The maximum growth rate occurs for $l = 0$ and $H \approx 0.8 H_R$, i.e., $1/k \approx 1.2a$ where a is the Rossby radius.