Fluid Dynamics of the Atmosphere and Ocean, ECMM719, 2018. Problem Set 1 (Assessed)

This problem set counts 10% toward your final mark. It is due on 5 November, 2018. The marks available for each question are in the right margin, and you should attempt all questions.

Partial credit will be given if the algebra is wrong but the method right, *provided you explain what you are trying to do.* Mathematical communication counts for 10% of the marks, in this and in all assessments, and in the final.

1. *(a)* The formula

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \boldsymbol{A}$$

relates the total rate of change of a vector in an inertial frame (subscript I) to that in a frame rotating with angular velocity $\boldsymbol{\Omega}$ (subscript R). Obtain an expression for

$$\left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{I}}$$

in terms of rotating frame quantities. If $\boldsymbol{A} = \boldsymbol{r}$, where $\boldsymbol{r}$ is the position vector, interpret your expression and identify the Coriolis and centrifugal terms

(b) Show that

$$\left(\frac{d^3\boldsymbol{A}}{dt^3}\right)_{\mathrm{I}} = \left(\frac{d^3\boldsymbol{A}}{dt^3}\right)_{\mathrm{R}} + 3\boldsymbol{\Omega} \times \left(\frac{d^2\boldsymbol{A}}{dt^2}\right)_{\mathrm{R}}$$
$$+ 3\boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathrm{R}}\right) - |\boldsymbol{\Omega}|^2\, \boldsymbol{\Omega} \times \boldsymbol{A}.$$

{20}

2. *(a)* Consider a fluid that obeys the hydrostatic relation

$$\frac{\partial p}{\partial z} = -\rho g.$$

Suppose also that the fluid is an isothermal ideal gas. Show that the density and pressure both diminish exponentially with height. What is the e-folding height? (This is also called the 'scale height' of the atmosphere.) Write down an expression for the height, z, as a function of pressure.

(b) Now suppose that the atmosphere has a uniform lapse rate (i.e., $\mathrm{d}T/\mathrm{d}z = -\Gamma$ = constant). Show that the height at a pressure p is given by

$$z = \frac{T_0}{\Gamma}\left[1 - \left(\frac{p_0}{p}\right)^{-R\Gamma/g}\right]$$

where T_0 is the temperature at $z = 0$.

(c) Are the answers you obtained in these two parts the same as each other in the isothermal (constant temperature) limit? Explain your answer. {20}

3. (a) In an adiabatic shallow water fluid in a rotating reference frame show that the potential vorticity conservation law is

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{\eta - h_b} = 0$$

where η is the height of the free surface and h_b is the height of the bottom topography, both referenced to the same flat surface.

(b) An air column at 60° N with zero relative vorticity ($\zeta = 0$) stretches from the surface to the tropopause, which we assume is a rigid lid, at 10 km. The air column moves zonally on to a plateau 2.5 km high. What is its relative vorticity? Suppose it then moves southwards to 30° N, staying on the plateau. What is its relative vorticity then? (Assume that the density is constant.) {15}

4. The shallow water equations, linearized about a state of rest, may be written as

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial \eta'}{\partial x}, \qquad \frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial \eta'}{\partial y},$$

$$\frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0.$$

Suppose there is a solid boundary (e.g., a coastline) at $x = 0$, with the ocean on one side and land on the other. Look for solutions that have $u' = 0$ everywhere, and with $f_0 > 0$. Show that the resulting waves are non-dispersive and travel at a speed $c = \sqrt{gH}$. Is the coastline to the left or to the right of the direction of travel? Suppose that these waves are generated just off the shore of North Carolina. Do they move north or south? Explain your answer. {20}

5. **Geostrophic adjustment with a velocity jump.**

(a) Show that the *linearized* potential vorticity, q', for the shallow water system is given by

$$q' = \zeta' - f_0\frac{h'}{H},$$

using standard notation.

(b) If the flow is in geostrophic balance show that the relative vorticity is given by

$$\zeta' = \nabla^2\psi,$$

where $\psi = gh'/f_0$. Hence show that the potential vorticity is then given by

$$q' = \nabla^2\psi - \frac{1}{L_d^2}\psi,$$

and write down an expression for L_d. (If you wish you may take $g = 1$.)

(c) Suppose that the initial flow has a flat surface with $u' = 0$ everywhere and Suppose that initially the fluid surface is flat, the zonal velocity (u) is zero and the meridional velocity is given by

$$v(x) = v_0 \operatorname{sgn}(x)$$

(i) Find the equilibrium height and velocity fields at $t = \infty$, in the linear approximation.

(ii) What are the initial and final kinetic and potential energies?

Hint: The potential vorticity is $q = \zeta - f_0\eta/H$, so that the initial potential vorticity is given by

$$q = 2v_0\,\delta(x).$$

where $\delta(x)$ is the Dirac delta function. {25}