# Fluid Dynamics of the Atmosphere and Ocean, ECMM719, 2018
## Problem Set 2 (Assessed)

This problem set counts 10% toward your final mark. It is due on 14 December, 2018. Full marks can be achieved by answering 5 of the 6 questions. All questions have equal credit.

Partial credit will be given if the algebra is wrong but the method right, *provided you explain what you are trying to do.* Mathematical communication counts for 10% of the marks.

1. **Geostrophic Theory**

   (a) In non-rotating flow show that the mass continuity equation, $\nabla \cdot \boldsymbol{v} = 0$, implies that the vertical velocity $W$ scales like $W = UH/L$ (in our usual notation). For large scale flow in the atmosphere show that vertical velocity will smaller than the horizontal velocity. Does this result hold in clouds? Explain your answer. {6}

   (b) Consider 3D flow in the Boussinesq equations at small Rossby number on the beta plane, with $f = f_0 + \beta y$. Using the equation of geostrophic balance, obtain a scaling for the magnitude of the vertical velocity in terms of $U$, $H$, $\beta$, $f$ and $L$. Then show that if $|\beta y| \ll |f_0|$ then the vertical velocity can be expected to be *smaller* than a scaling based solely on the mass continuity equation (as in part (a)) for large-scale flow. {7}

   (c) Consider the shallow water planetary-geostrophic equations. Show that if $f$ is constant (i.e., $f = f_0$) then the there is no evolution in the height equation. That is, in the height equation,
$$\frac{\partial h}{\partial t} + \boldsymbol{u} \cdot \nabla h + h \nabla \cdot \boldsymbol{u} = 0,$$
   show that $\partial h/\partial t = 0$. (Bear in mind that the flow is geostrophically balanced.) {7}

   **{20}**

2. **Vertically propagating Rossby waves**

   (a) The quasigeostrophic potential vorticity equation, linearized about a constant eastward mean flow $U$ is
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) q + \beta v = 0,$$
   where
$$q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)$$
   is the perturbation potential vorticity. Here, $\psi$ is the perturbation geostrophic stream function, $z$ is a vertical coordinate, $f_0$, $N$ and $\beta > 0$ are constants.
   Derive the dispersion relation
$$\omega = kU - \frac{k\beta}{k^2 + l^2 + m^2\left(f_0^2/N^2\right)}.$$
   What kind of waves are these? {7}

   (b) Calculate the vertical component of the group velocity for such waves. {4}

   (c) The perturbation $v$ and $b$ components will also have wavelike behaviour
$$(v, b) = \mathrm{Re}\left\{(\hat{v}, \hat{b}) \exp\left[\mathrm{i}\left(kx + ly + mz - \omega t\right)\right]\right\},$$

where $b = f_0 \partial\psi/\partial z$. The northward eddy bouyancy flux is

$$\overline{vb} \equiv \left(\frac{1}{L_\lambda}\right) \int_{L_\lambda} v\, b\, dx$$

where $L_\lambda$ is one wavelength. Show that

$$\overline{vb} = Akm\, |\hat{\psi}|^2,$$

where $A$ is a constant. What is it? Hence explain why upward wave propagation requires a poleward eddy bouyancy flux. {5}

(d) Now consider *stationary* waves. Show that upward wave propagation is possible only if $U$ satisfies the inequality

$$\frac{\beta}{k^2 + l^2} > U > 0.$$

{4}

**{20}**

3. **Ekman layers**

Consider a layer of fluid of constant density at in the upper ocean that satisfies the *Ekman-layer* equations:

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau_x}{\partial z}, \qquad fu = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau_y}{\partial z}. \qquad \text{(Ek)}$$

where $\tau_x, \tau_y$ are components of the stress, $\boldsymbol{\tau}$, in the $x$- and $y$-directions and $f = f_0 + \beta y$. Assume that the pressure, $\phi$, is not a function of $z$, that the Ekman layer has some finite depth, $H_E$, below which the stress is zero, and that the vertical velocity is zero at the top of the ocean, $z = 0$, and at the bottom, $z = -H_o$.

(a) Show that the total ageostrophic transport induced by the stress is at right angles to the direction of the surface stress.

(b) By integrating the mass continuity equation over the entire depth of the ocean show that the vertically integrated geostrophic velocity is given by

$$\int_{-H_o}^{0} \beta v_g \,\mathrm{d}z = f\left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right]. \qquad \text{(A)}$$

(c) By cross-differentiating equations (Ek) and vertically integrating over the total depth of the ocean, or otherwise, derive the Sverdrup relation,

$$\int_{-H_o}^{0} \beta v \,\mathrm{d}z = \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y} \qquad \text{(B)}$$

where $v$ is the meridional component of the total velocity.

(d) If $f$ is not constant eq. (A) is different from eq. (B). Explain how and why the two equations are nevertheless consistent. **{20}**

4. **Rossby waves and jets**

(a) Show that if there is a source of Rossby waves at any given latitude in the Northern Hemisphere, we expect that eastward flow will be generated there. Your answer will involve relating flux of momentum in Rossby waves and relating it to group velocity. How does your answer differ in the Southern Hemisphere?

(b) Consider two interacting Rossby waves in a single-layer barotropic fluid. Each Rossby wave generates a wave with a velocity amplitude of $10\,\mathrm{m\,s^{-1}}$. Being explicit about the assumptions you make, what is the acceleration of the mean flow? How long will it take to generate a mean flow of $20\,\mathrm{m\,s^{-1}}$? **{20}**

5. **More Geostrophic Theory**

(a) Begin with the shallow water potential vorticity equation,

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{h} = 0 \qquad \text{(SWPV)}$$

where $\zeta$ is the relative vorticity, $h$ is the height field and $f = f_0 + \beta y$, where $|\beta y| \ll f_0$. By supposing that the flow is nearly in geostrophic balance, and that the perturbations in the height field are small (that is, $h = H + \eta$ where $H$ is a constant and $|\eta| \ll H$) derive the *quasi-geostrophic* potential vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi - k_d^2\psi\right) + \beta v = 0,$$

where $\psi$ is the streamfunction and $\nabla^2 = \partial_x^2 + \partial_y^2$. What is $k_d$? [8]

(b) The *planetary geostrophic* equations may be derived by simply omitting $\zeta$ in equation (SWPV), by invoking a small Rossby number, so that $\zeta/f$ is small. We then relate the velocity field to the height field by hydrostatic balance and obtain:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{f}{h}\right) = 0, \qquad fu = -g\frac{\partial h}{\partial y}, \quad fv = g\frac{\partial h}{\partial x}.$$

The assumption of small Rossby number are the same as those used in deriving the quasi-geostrophic equations of Part (i). Explain carefully how the derivations nevertheless differ, and how the assumptions used for quasi-geostrophy are in fact different from those used in planetary-geostrophy. Use any or all of the momentum and mass continuity equations, scaling, nondimensionalization and verbal explanations as needed. {12}

**{20}**

6. **Western boundary layers**

Consider the barotropic vorticity equation in the form

$$\frac{\partial\zeta}{\partial t} + J(\psi,\zeta) + \beta v = -r\zeta + \nu\nabla^2\zeta + F(x,y),$$

where $\zeta = \nabla^2\psi$, $v = \partial\psi/\partial x$ and $r$ and $\nu$ are constants, and the flow is two-dimensional. We suppose the fluid is contained in a square container of side $a$, with $0 \le x \le a$ and $0 \le x \le a$ and $F = -A\sin\pi y/a$ where $A$ is a constant. We expect that the nonlinear term and both frictional terms are 'small', and we are interested in steady states for which $\partial\zeta/\partial t = 0$.

(a) Nondimensionalize these equations, and obtain estimates of the sizes of each term. State explicitly the conditions under which each of the frictional terms, and the nonlinear term, are indeed small.

(b) Neglecting the nonlinear term and both frictional terms, obtain the solution to $\beta\partial\psi/\partial x = -A\sin\pi y/a$. Can this solution satisfy the boundary conditions needed if the frictional terms are present. Explain briefly.

(c) Suppose that $\nu = 0$, and neglect the nonlinear term, and assume that the term $r\zeta$ is indeed small but nonzero. Show that we can expect a boundary current on one side of the ocean (which?) and estimate its thickness.

(d) Now suppose that $r = 0$, and neglect the nonlinear term. Estimate the size of the boundary current that now arises. **{20}**