ECMM719U/P

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

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Fluid Dynamics of Atmospheres and Oceans

Module Leader: Prof. Geoffrey Vallis

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Approximately 10% of the marks will be given for mathematical communication; that is, marks will be awarded, or subtracted, for clarity and style of explanation. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) The horizontal momentum equation for a Boussinesq fluid may be written as

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{\partial\phi}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{\partial\phi}{\partial y}$$

Define the Rossby number, and show that if the Rossby number is small then the flow can be expected to be close to geostrophic balance. Suppose that the flow is in hydrostatic balance, which we write as

$$\frac{\partial\phi}{\partial z} = b. \tag{A}$$

where b is the buoyancy, which you may think of as temperature. By combining geostrophic balance with equation (A), show that a horizontal gradient of buoyancy is associated with a vertical shear. [11]

(b) The shallow water equations may be written as

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0,$$

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0,$$
$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0,$$

in Cartesian coordinates (x, y). Here u and v are the velocity components, h is the layer thickness, f is the Coriolis parameter, and g is a constant. Derive the energy conservation law

$$\frac{\partial E}{\partial t} + \frac{\partial F^{(x)}}{\partial x} + \frac{\partial F^{(y)}}{\partial y} = 0,$$

where

$$E = g\frac{h^2}{2} + h\frac{u^2+v^2}{2},$$

and give explicit expressions for the components of the energy flux $(F^{(x)}, F^{(y)})$. [15]

(c) In a rotating frame of reference the rate of change of a vector $\boldsymbol{B}$ in an inertial frame is related to its rate of change in the rotating frame by the formula

$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{B}. \tag{B}$$

Use this relation to obtain an expression for the second derivative, namely

$$\left(\frac{\mathrm{d}^2\boldsymbol{B}}{\mathrm{d}t^2}\right)_I$$

in terms of rotating frame quantities. If $\boldsymbol{B} = \boldsymbol{r}$ then identify the Coriolis force and the centrifugal force in your expression and briefly give a physical interpretation.

Can we apply equation (B) directly to velocity? That is, is it correct to say that the acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ in the rotating frame and in the inertial frame are related by

$$\boldsymbol{a}_I = \boldsymbol{a}_R + \boldsymbol{\Omega} \times \boldsymbol{v}.$$

Explain your answer. [12]

(d) Consider the vertical momentum equation in the form

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \qquad \text{(C)}$$

Is it always correct to say that the hydrostatic approximation is appropriate when the vertical acceleration is much less than g? That is, when

$$\left|\frac{\mathrm{D}w}{\mathrm{D}t}\right| \ll g.$$

Explain your answer. In general, under what circumstances does hydrostatic balance hold?

Suppose that the density is constant with $\rho = \rho_0$. Show that equation (C) can be written in the form

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}.$$

where $p' = p + \rho_0 g z$. Can the hydrostatic approximation be valid here? Briefly explain. [12]

[50]

End of Part A

SECTION B

2. (a) Begin with the shallow water potential vorticity equation,

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{h} = 0$$

where ζ is the relative vorticity, h is the height field and $f = f_0 + \beta y$, where $|\beta y| \ll f_0$. By supposing that the flow is nearly in geostrophic balance, and that the perturbations in the height field are small (that is, $h = H + \eta$ where H is a constant and $|\eta| \ll H$) derive the quasi-geostrophic potential vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi - k_d^2\psi\right) + \beta v = 0,$$

where ψ is the streamfunction and $\nabla^2 = \partial_x^2 + \partial_y^2$. What is k_d? [9]

(b) Let $k_d = 0$ and linearize the system about a state of rest. By considering perturbations of the form

$$\psi = \mathrm{Re}\left\{\Psi \exp\left[\mathrm{i}(kx + ly - \omega t)\right]\right\},$$

or otherwise, show that the dispersion relation for this system is

$$\omega = -\frac{\beta k}{k^2 + l^2},$$

and hence obtain an expression for the y–component of the group velocity. [7]

(c) The meridional component of the eddy momentum flux (per unit mass) is given by:

$$\overline{uv} = \frac{1}{L}\int_L uv\,\mathrm{d}x = \frac{1}{L}\int_L \left(-\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\psi}{\partial x}\right)\mathrm{d}x,$$

where L is one wavelength. Using this, show that

$$\overline{uv} = -\frac{1}{2}kl|\Psi|^2.$$

Hence infer that the meridional component of the group velocity has the opposite sign to the momentum flux. Briefly explain how this can produce westerly jets in midlatitude atmospheres. [9]

[**25**]

3. Consider a layer of fluid of constant density in the upper ocean that satisfies the Ekman-layer equations:

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau_x}{\partial z}, \qquad fu = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau_y}{\partial z}, \qquad \text{(E)}$$

where τ_x, τ_y are components of the stress, $\boldsymbol{\tau}$, in the x- and y-directions and $f = f_0 + \beta y$. Assume that the pressure, ϕ, is not a function of z, that the Ekman layer has some finite depth, H_E, below which the stress is zero, and that the vertical velocity is zero at the top of the ocean, $z = 0$, and at the bottom.

(a) Define the geostrophic velocity, (u_g, v_g), in terms of the components of the pressure. Show that the divergence of the geostrophic velocity satisfies

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = -\beta v_g.$$

Show also that equations (E) may be written as

$$f(v_g - v) = \frac{\partial \tau_x}{\partial z}, \qquad f(u - u_g) = \frac{\partial \tau_y}{\partial z}. \qquad \text{(F)}$$

[6]

(b) Suppose that the stress is imposed at the top of the layer ($z = 0$) such that

$$\tau_x = \tau_{x0}, \quad \tau_y = \tau_{y0} \quad \text{at} \quad z = 0.$$

At the bottom of the Ekman layer suppose that the stress is zero.

By integrating equations (F) over the depth of the Ekman layer show that the transport induced by the stress (i.e., the ageostrophic mass flux) is at right angles to the direction of the surface stress. [6]

(c) By integrating the mass continuity equation over the depth of the Ekman layer show that the vertical velocity at the base of the Ekman layer, w_E, is given by

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta}{f} v_g \,\mathrm{d}z$$

[7]

(d) By cross-differentiating equations (E) and vertically integrating over the total depth of the ocean, or otherwise, derive the Sverdrup relation,

$$\int \beta v \,\mathrm{d}z = \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y}$$

where v is the meridional component of the total velocity (i.e. geostrophic and ageostrophic). [6]

[**25**]

4. (a) A planet rotates with angular velocity Ω. Write down an expression for the absolute angular momentum of a fluid parcel at a distance r from the centre of the planet and latitude ϕ with relative velocity u.

An air parcel is initially at rest relative to the rotating Earth at the surface on the equator. Calculate its absolute angular momentum per unit mass. (Assume the Earth to be spherical with radius $a = 6.4 \times 10^6\,\mathrm{m}$ and $\Omega = 7.292 \times 10^{-5}\,\mathrm{s}^{-1}$.) [5]

(b) The air parcel rises to a height of 10 km while conserving its absolute angular momentum. What is the velocity u acquired by the the parcel and in what direction is this? The parcel then moves to 30° N at the same height, again conserving its absolute angular momentum. What is its final value of u, and in what direction is the flow? [7]

(c) Suppose that at the ground the velocity of the air is zero, and that it increases linearly to its value at 10 km at 30° N, as calculated above. Suppose also that the flow obeys the thermal wind relation in the form

$$f \frac{\partial u}{\partial z} = -\frac{g}{T_0} \frac{\partial T}{\partial y},$$

where $f = 2\Omega \sin \phi$ where ϕ is latitude, T is temperature, $T_0 = 300\,\mathrm{K}$ and $g = 10\,\mathrm{m\,s^{-2}}$. Calculate the meridional temperature gradient at 30° N. Using this value, or otherwise, estimate the temperature fall off between the equator and 30° N. (You may assume ϕ is small and $\sin \phi \approx \phi$.) [8]

(d) Are the values you obtained in parts (b) and (c) realistic for the Earth's atmosphere? Explain you answer, and discuss the relevance of this to the extent of Earth's Hadley circulation. [5]

[In your answers above, clarity of explanation and the correct methodology carry more weight than the correct numerical answers.] [**25**]