ECMM719U/P

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

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Fluid Dynamics of Atmospheres and Oceans

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Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the questions in Section B (25% for each).

Marks shown in the margins, such as [7], are a guideline. Approximately 10% of the marks will be given for mathematical communication; that is, marks will be awarded for clarity and style of explanation. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. a The horizontal momentum equation for a Boussinesq fluid may be written as

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{\partial\phi}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{\partial\phi}{\partial y}$$

Define the Rossby number, and show that if the Rossby number is small then the flow can be expected to be close to geostrophic balance. Suppose that the flow is in hydrostatic balance, which we write as

$$\frac{\partial\phi}{\partial z} = b. \tag{A}$$

where b is the buoyancy, which you may think of as temperature. By combining geostrophic balance with equation (A), show that a horizontal gradient of buoyancy is associated with a vertical shear. [11]

------------------------- *Begin solution* -------------------------

The Rossby number is $Ro \equiv U/(fL)$ where U is a scale for the horizontal velocity and L a horizontal length scale.

If Roł1 then the Coriolis term is much larger than the material derivative and must be balanced by the pressure gradient, and this is geostrophic balance.

Take the vertical derivative of geostrophic balance:

$$-f\frac{\partial v}{\partial z} = -\frac{\partial}{\partial z}\frac{\partial\phi}{\partial x}, \qquad f\frac{\partial u}{\partial z} = -\frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}$$

Then use (A) to give

$$f\frac{\partial v}{\partial z}\frac{\partial b}{\partial x}, \qquad f\frac{\partial u}{\partial z} = -\frac{\partial b}{\partial y}$$

Hence a horizontal gradient of b gives rise to a vertical gradient of $\boldsymbol{u}$.

------------------------- *End Solution* -------------------------

b The shallow water equations may be written as

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0,$$

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0,$$

$$\frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0,$$

in Cartesian coordinates (x, y). Here u and v are the velocity components, h is the layer thickness, f is the Coriolis parameter, and g is a constant. Derive the energy conservation law

$$\frac{\partial E}{\partial t} + \frac{\partial F^{(x)}}{\partial x} + \frac{\partial F^{(y)}}{\partial y} = 0,$$

where

$$E = g\frac{h^2}{2} + h\frac{u^2 + v^2}{2},$$

and give explicit expressions for the components of the energy flux $(F^{(x)}, F^{(y)})$.

[15]

------------------------- *Begin solution* -------------------------

Multiply the height equation by h and the momentum equations by $h\boldsymbol{u}$ to give

$$\frac{\partial}{\partial t}\frac{h^2}{2} + \nabla\cdot\left(\boldsymbol{u}\frac{h^2}{2}\right) + \frac{h^2}{2}\nabla\cdot\boldsymbol{u} = 0,$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{h\boldsymbol{u}^2}{2} + \frac{\boldsymbol{u}^2 h}{2}\nabla\cdot\boldsymbol{u} = -g\boldsymbol{u}\cdot\nabla\frac{h^2}{2} \tag{1}$$

or

$$\frac{\partial}{\partial t}\frac{h\boldsymbol{u}^2}{2} + \nabla\cdot\left(\boldsymbol{u}\frac{h\boldsymbol{u}^2}{2}\right) + g\boldsymbol{u}\cdot\nabla\frac{h^2}{2} = 0. \tag{2}$$

Adding these gives

$$\frac{\partial}{\partial t}\frac{1}{2}\left(h\boldsymbol{u}^2 + gh^2\right) + \nabla\cdot\left[\frac{1}{2}\boldsymbol{u}\left(gh^2 + h\boldsymbol{u}^2 + gh^2\right)\right] = 0, \tag{3}$$

or

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = 0, \tag{4}$$

where $E = KE + PE = (h\boldsymbol{u}^2 + gh^2)/2$ is the density of the total energy and $\boldsymbol{F} = \boldsymbol{u}(h\boldsymbol{u}^2/2 + gh^2)$ is the energy flux, and $\boldsymbol{u} = (u, v)$.

------------------------- *End Solution* -------------------------

c In a rotating frame of reference the rate of change of a vector $\boldsymbol{B}$ in an inertial frame is related to its rate of change in the rotating frame by the formula

$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}. \tag{B}$$

Use this relation to obtain an expression for the second derivative, namely

$$\left(\frac{\mathrm{d}^2\boldsymbol{B}}{\mathrm{d}t^2}\right)_I$$

in terms of rotating frame quantities. If $\boldsymbol{B} = \boldsymbol{r}$ then identify the Coriolis force and the centrifugal force in your expression and briefly give a physical interpretation.

Can we apply equation (B) directly to velocity? That is, is it correct to say that the acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ in the rotating frame and in the inertial frame are related by

$$\boldsymbol{a}_I = \boldsymbol{a}_R + \boldsymbol{\Omega} \times \boldsymbol{v}.$$

Explain your answer. (12)

------------------------- *Begin solution* -------------------------

Apply (B) to itself to obtain

$$\left(\frac{\mathrm{d}^2\boldsymbol{B}}{\mathrm{d}t^2}\right)_I = \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\right)\left(\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}\right).$$

or

$$\left(\frac{\mathrm{d}^2\boldsymbol{B}}{\mathrm{d}t^2}\right)_I = \left(\frac{\mathrm{d}^2\boldsymbol{B}}{\mathrm{d}t^2}\right)_R + 2\boldsymbol{\Omega} \times \left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_R + \Omega \times \Omega \times \boldsymbol{B}.$$

If $\boldsymbol{B} = \boldsymbol{r}$ then the second and third terms on the RHS are the Coriolis and centrifugal terms, respectively.

We cannot apply this to velocity because velocity is not measured to be the same in the rotating and inertial frames.

------------------------- *End Solution* -------------------------

d Consider the vertical momentum equation in the form

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{C}$$

Is it always correct to say that the hydrostatic approximation is appropriate when the vertical acceleration is much less than g? That is, when

$$\left|\frac{\mathrm{D}w}{\mathrm{D}t}\right| \ll g.$$

Explain your answer. In general, under what circumstances does hydrostatic balance hold?

Suppose that the density is constant with $\rho = \rho_0$. Show that equation (C) can be written in the form

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}.$$

where $p' = p + \rho_0 g z$. Can the hydrostatic approximation be valid here? Briefly explain. [12]

------------------------ *Begin solution* ------------------------

It is not always correct to say this. We need for the vertical acceleration to be smaller than g', where $g' = g\delta\rho/\rho'$. Thus, if $\delta\rho = 0$, as in a constant density fluid, the hydrostatic balance doesn't normally hold.

If we write $\rho = \rho_0 + \rho'$and $p = p_0 + p'$, where $p_0 = -\rho_0 g z$, then (C) becomes

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}.$$

The gravity term now has no dynamical effect and the hydrostatic approximation is not really meaningful.

------------------------ *End Solution* ------------------------

[50]

End of Part A

SECTION B

2. a Begin with the shallow water potential vorticity equation,

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{h} = 0$$

where ζ is the relative vorticity, h is the height field and $f = f_0 + \beta y$, where $|\beta y| \ll f_0$. By supposing that the flow is nearly in geostrophic balance, and that the perturbations in the height field are small (that is, $h = H + \eta$ where H is a constant and $|\eta| \ll H$) derive the quasi-geostrophic potential vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi - k_d^2\psi\right) + \beta v = 0,$$

where ψ is the streamfunction and $\nabla^2 = \partial_x^2 + \partial_y^2$. What is k_d? [9]

------------------------- *Begin solution* -------------------------

If the flow is nearly in geostrophic balance then the flow is non-divergent and $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial z$ where $\psi = g\eta/f_0$. Also, $\zeta = \nabla^2\psi$. Now, the shallow water potential vorticity may be approximated as

$$Q = \frac{\zeta + f}{h} = \frac{1}{H}\frac{\zeta + f}{1 + \eta/H} \approx \frac{1}{H}(\zeta + f)(1 - \eta/H)$$
$$\approx \frac{1}{H}\left(f_0 + \zeta + \beta y - f_0\eta/H\right),$$

dropping the smallest terms. The term f_0/H is dynamically unimportant, as is the constant factor of H on the other terms, so that the dynamically important part of PV is

$$q = \zeta + \beta y - f_0\frac{\eta}{H} = \nabla^2\psi - \frac{f_0^2\psi}{gH}.$$

We thus recover the answer given with $k_d = f_0/\sqrt{gH}$.

------------------------- *End Solution* -------------------------

b Let $k_d = 0$ and linearize the system about a state of rest. By considering perturbations of the form

$$\psi = \mathrm{Re}\left\{\Psi \exp\left[\mathrm{i}(kx + ly - \omega t)\right]\right\},$$

or otherwise, show that the dispersion relation for this system is

$$\omega = -\frac{\beta k}{k^2 + l^2},$$

and hence obtain an expression for the y–component of the group velocity. [7]

------------------------- *Begin solution* -------------------------

The linear equation is

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x}.$$

Substituting

$$\psi = \mathrm{Re}\,\{\Psi \exp\left[\mathrm{i}(kx + ly - \omega t)\right]\},$$

immediately gives the answer. The y–component of the group velocity is given by

$$c_g^2 = \frac{\partial \omega}{\partial y} = \frac{2\beta k l}{(k^2 + l^2)^2}$$

------------------------- *End Solution* -------------------------

c The meridional component of the eddy momentum flux (per unit mass) is given by:

$$\overline{uv} = \frac{1}{L}\int_L uv\,\mathrm{d}x = \frac{1}{L}\int_L \left(-\frac{\partial \psi}{\partial y}\right)\left(\frac{\partial \psi}{\partial x}\right)\mathrm{d}x,$$

where L is one wavelength. Using this, show that

$$\overline{uv} = -\frac{1}{2}kl|\Psi|^2.$$

Hence infer that the meridional component of the group velocity has the opposite sign to the momentum flux. Briefly explain how this can produce westerly jets in midlatitude atmospheres. [9]

------------------------- *Begin solution* -------------------------

The harmonic forms of u and v are

$$u' = -\mathrm{Re}\,C\,\mathrm{i}l\,\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \qquad v' = \mathrm{Re}\,C\,\mathrm{i}k\,\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{5}$$

where C is a constant. The associated momentum flux is

$$\overline{u'v'} = -\frac{1}{2}C^2kl. \tag{6}$$

which is of opposite direction to the group velocity. Hence a source of Rossby waves is associated with a convergence of momentum.

- *End Solution* -

[**25**]

3. Consider a layer of fluid of constant density in the upper ocean that satisfies the Ekman-layer equations:

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau_x}{\partial z}, \qquad fu = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau_y}{\partial z}, \qquad \text{(E)}$$

where τ_x, τ_y are components of the stress, $\boldsymbol{\tau}$, in the x- and y-directions and $f = f_0 + \beta y$. Assume that the pressure, ϕ, is not a function of z, that the Ekman layer has some finite depth, H_E, below which the stress is zero, and that the vertical velocity is zero at the top of the ocean, $z = 0$, and at the bottom.

a Define the geostrophic velocity, (u_g, v_g), in terms of the components of the pressure. Show that the divergence of the geostrophic velocity satisfies

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = -\beta v_g.$$

Show also that equations (E) may be written as

$$f(v_g - v) = \frac{\partial \tau_x}{\partial z}, \qquad f(u - u_g) = \frac{\partial \tau_y}{\partial z}. \qquad \text{(F)}$$

[6]

------------------------- *Begin solution* -------------------------

The geostrophic velocity is defined to be such that

$$-fv_g = -\frac{\partial \phi}{\partial x} \qquad fu_g = -\frac{\partial \phi}{\partial y}$$

Cross differentiate the above to give

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = -\beta v_g.$$

where $\beta = \partial f/\partial y$. Substituting the defintion of the geostrophic velocity into equation (E) gives

$$f(v_g - v) = \frac{\partial \tau_x}{\partial z}, \qquad f(u - u_g) = \frac{\partial \tau_y}{\partial z}.$$

------------------------- *End Solution* -------------------------

b Suppose that the stress is imposed at the top of the layer ($z = 0$) such that

$$\tau_x = \tau_{x0}, \quad \tau_y = \tau_{y0} \qquad \text{at} \qquad z = 0.$$

At the bottom of the Ekman layer suppose that the stress is zero.
By integrating equations (F) over the depth of the Ekman layer show that the transport induced by the stress (i.e., the ageostrophic mass flux) is at right angles to the direction of the surface stress. [6]

------------------------- *Begin solution* -------------------------

Integrate (F) from the top of the ocean to the bottom of the Ekman layer to give

$$\int f v_E \, \mathrm{d}z = -\tau_x(0), \qquad \int f v_E \, \mathrm{d}z = \tau_y z(0).$$

where $\boldsymbol{u}_E = (u - u_g, v - v_g)$ is the ageostrophic Ekman velocity. From the above it is clear that $\int \boldsymbol{u}_E \, \mathrm{d}z$ is at right angles to the stress.

-------------------------- *End Solution* --------------------------

c By integrating the mass continuity equation over the depth of the Ekman layer show that the vertical velocity at the base of the Ekman layer, w_E, is given by

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta}{f} v_g \, \mathrm{d}z$$

[7]

------------------------- *Begin solution* -------------------------

The mass continuity equation is $\partial w/\partial z = -\nabla \cdot \boldsymbol{u}$. Start with (E), divide through by f and then take the divergence to give

$$\nabla \cdot \boldsymbol{u} = -\frac{\beta}{f^2}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}\frac{1}{f}\frac{\partial \tau_y}{\partial z} - \frac{\partial}{\partial y}\frac{1}{f}\frac{\partial \tau_x}{\partial z}$$

The first term on the rhs is $\beta v_g / f$. Integrating over the depth of the Ekmnan layer gives, using the mass continuity equation,

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta}{f} v_g \, \mathrm{d}z$$

-------------------------- *End Solution* --------------------------

d By cross-differentiating equations (E) and vertically integrating over the total depth of the ocean, or otherwise, derive the Sverdrup relation,

$$\int \beta v \, \mathrm{d}z = \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y}$$

where v is the meridional component of the total velocity (i.e. geostrophic and ageostrophic). [6]

------------------------- *Begin solution* -------------------------

If we begin (E) and cross differentiate (without dividing by f first) we obtain

$$f \nabla \cdot \boldsymbol{u} + \beta v = \frac{\partial}{\partial x} \frac{\partial \tau_y}{\partial z} - \frac{\partial}{\partial y} \frac{\partial \tau_x}{\partial z}$$

If we integrate over the full depth of the ocean the divergence term on the lhs cancels and we obtain

$$\int \beta v \, \mathrm{d}z = \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y}$$

------------------------- *End Solution* -------------------------

[25]

4. a A planet rotates with angular velocity Ω. Write down an expression for the absolute angular momentum of a fluid parcel at a distance r from the centre of the planet and latitude ϕ with relative zonal velocity u. An air parcel is initially at rest relative to the rotating Earth at the surface on the equator. Calculate its absolute angular momentum per unit mass. (Assume the Earth to be spherical with radius $a = 6.4 \times 10^6$ m and $\Omega = 7.292 \times 10^{-5}\,\mathrm{s}^{-1}$.) [5]

------------------------- *Begin solution* -------------------------

$$M_a = (\Omega r^2 \cos\phi + ur)\cos\phi = (\Omega r \cos\phi + u) r \cos\phi$$

On the Earth at the equator at rest we have

$$M_a = \Omega a^2 = 7.292 \times 10^{-5} a^2 = 2.98 \times 10^9\,\mathrm{m}^2\,\mathrm{s}^{-1}$$

per unit mass.

-------------------------- *End Solution* --------------------------

b The air parcel rises to a height of 10 km while conserving its absolute angular momentum. What is the velocity u acquired by the the parcel and in what direction is this? Finally the parcel moves to 30° N at the same height, again conserving its absolute angular momentum. What is its final value of u?, and in what direction is the flow? [7]

------------------------- *Begin solution* -------------------------

The final value of u is given by solving

$$(\Omega a \cos\phi + 0) a \cos\phi = (\Omega(a+h)\cos\phi + u)(a+h)\cos\phi \qquad (7)$$

where h is 10 km and $\cos\phi = 1$ (at equator). Thus, to a good approximation,

$$u(a+h) = -2\Omega a h - \Omega h^2 \qquad \text{giving} \qquad u \approx -2\Omega h$$

Putting in numbers gives $u = -0.15$ m.

Moving to 30° N we calculate the velocity there using

$$(\Omega a \cos 30 + u)\cos 30 = \Omega a$$

Or

$$u = \frac{\Omega a}{\cos 30}(1 - \cos^2 30) = \frac{\Omega a \sin^2 30}{\cos 30} = 269 m\,\mathrm{s}^{-1}.$$

-------------------------- *End Solution* --------------------------

c Suppose that at the ground the velocity of the air is zero, and that it increases linearly to its value at 10 km at 30° N, as calculated above. Suppose also that the flow obeys the thermal wind relation in the form

$$f \frac{\partial u}{\partial z} = -\frac{g}{T_0} \frac{\partial T}{\partial y},$$

where $f = 2\Omega \sin\phi$ where ϕ is latitude, T is temperature, $T_0 = 300$ K and $g = 10\,\mathrm{m\,s^{-2}}$. Calculate the meridional temperature gradient. Using this value, or otherwise, estimate the temperature fall off between the equator and 30° N. (You may assume ϕ is small and $\sin\phi \approx \phi$.) [8]

------------------------- *Begin solution* ------------------------- [8]

Let $f = 10^{-4}$ and $g = 10$. Then the temperature gradient is

$$\frac{\partial T}{\partial y} = \frac{f T_0}{g} \frac{\partial u}{\partial z} \approx \frac{10^{-4}\ 300}{10} \frac{269}{10^4} \approx 7 \times 10^{-5}\,\mathrm{K\,m^{-1}} \qquad (8)$$

or about 7 K per hundred kilomenters. This is too rapid - the temperature would fall by 70 degrees over 1000 kilometers.

------------------------- *End Solution* -------------------------

d Are the values you obtained in parts (b) and (c) realistic for the Earth's atmosphere? Explain you answer, and discuss the relevance of this to the extent of Earth's Hadley circulation. [5]

------------------------- *Begin solution* -------------------------

The value of the wind at the equator is reasonably okay (it is small). The value at 30° is too big - the flow is not in fact angular momentum conserving. And the temperature fall off is far too large.

------------------------- *End Solution* -------------------------

[**25**]