Six Easy Pieces

Concerning the Structure of the Atmosphere and Ocean

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Preface

October 13, 2014

These notes are an approximation of the lectures I gave at the GFD Summer School at Walsh Cottage in the summer of 2014. The content and level of coverage is very uneven – in some places I go into a lot of detail, elsewhere I skim over important concepts. In a few places these notes read like a book, but mostly they read like what they are, informal lecture notes. They will be updated again between now and the end of the year.

The lectures cover a fair bit of ground in the large-scale structure and circulation of the atmosphere and ocean and how they fit into the climate system, but there is no aim to be comprehensive. Most of the material is my take on matters that are floating around in the literature, such as the model of the runaway greenhouse effect in chapter 1, Rossby waves and momentum transport in chapter 3, and the model of the deep circulation of the ocean in chapter 6. A few aspects of the material give some new interpretation – the model of the tropopause height in chapter 2 and aspects of discussion of the Hadley Cell in chapter 4 for example. In any case, take it as you find it, and comments are always welcome.

Chapter 1

Energy Balance and the Troposphere

The philosophy throughout these lectures is that in order to understand a complex system we must have a description of the system at multiple levels, from a back-of-the-envelope calculation through idealized numerical models to a comprehensive simulation with all the bells and whistles. Because this is Walsh we will always try to include a back-of-the-envelope calculation and go from there, but other approaches are possible.

1.1 What are we trying to explain?

A schematic of the overall structure of the atmosphere and ocean is given in Fig. 1.1, with some pictures of the real atmosphere from observations given in Fig. 1.2 and Fig. 1.3. In Fig. 1.1 we sketch the troposphere, where temperature decreases with height, and the stratosphere, where temperature increases with height, and the dividing tropopause which is fairly high over then tropics (15km) and lower over polar regions (8km). We might immediately ask, what determines this structure? What determines the height of tropopause? Why is it about 10 km, and not 100 km or 1 km? And what determines the width of the tropics where the tropopause is high? And so on

Turning to the ocean, we have, again very schematically, warm water in the upper ocean and cold water below. The layer between them, where temperature varies very fast vertically, is called the thermocline. Sometimes we make an analogy between the thermocline and the tropopause, but actually the thermocline is more like the whole troposphere because they are both characterized by large vertical temperature gradient and relatively fast dynamics. Questions for oceanographers include what determines this structure of the ocean? What is the nature of the circulation that maintains it? More specifically, what determines the depth of the thermocline?

These are the *kinds* of questions we will consider in these lectures. We'll try to answer some of them, but not all. The philosophy throughout is that in order to understand a complex system we must have a description of the system at multiple levels, from a back-of-the-envelope calculation through idealized numerical models to a comprehensive simulation with all the

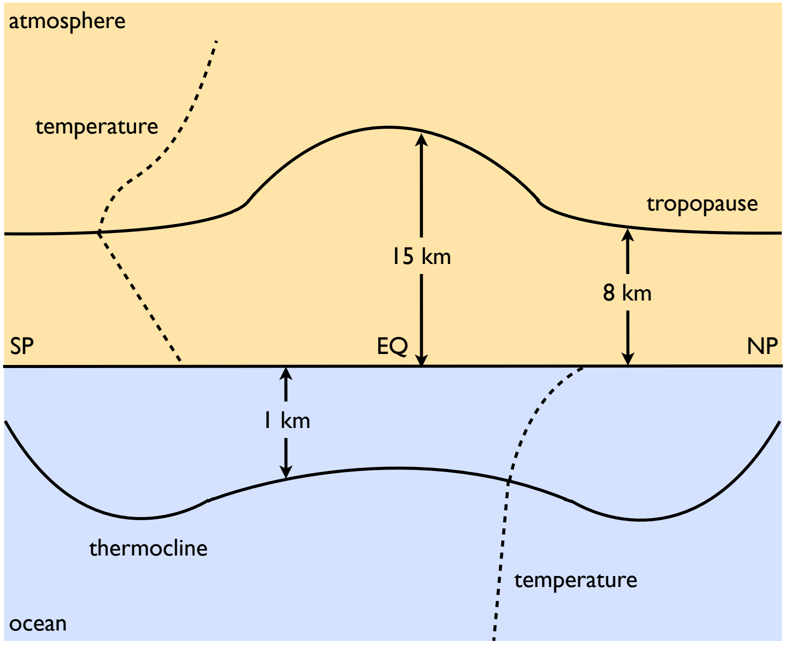


Figure 1.1 A schematic of thermal structures of ocean and atmosphere. The solid lines mark the tropopause and the base of the thermocline, and the near-vertical dashed lines are representative profiles of temperature.

bells and whistles. Because this is Walsh we will always try to include a back-of-the-envelope calculation and go from there, but other approaches are possible.

Our goals in this chapter are fairly fundamental:

1. Understand at an elementary level what determines the surface temperature of Earth.
2. Understand the need for a troposphere, and what determines its thickness.

To answer that we begin with a tutorial on radiation.

1.2 Radiative Balance

1.2.1 The very basics

All macroscopic bodies except those at absolute zero (are there any?) emit thermal radiation. The black body emission per unit wavelength or per unit frequency are given by Planck's function which is, for the two cases respectively,

$$B_\lambda(T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_B T) - 1}, \qquad B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T) - 1} \tag{1.1}$$

where c is the speed of light, h is the Planck constant and k_B is the Boltzmann constant. Conventions for frequency and wavelength are such that $c = \omega/k = \omega\lambda/2\pi = \nu\lambda$. Integrating either of

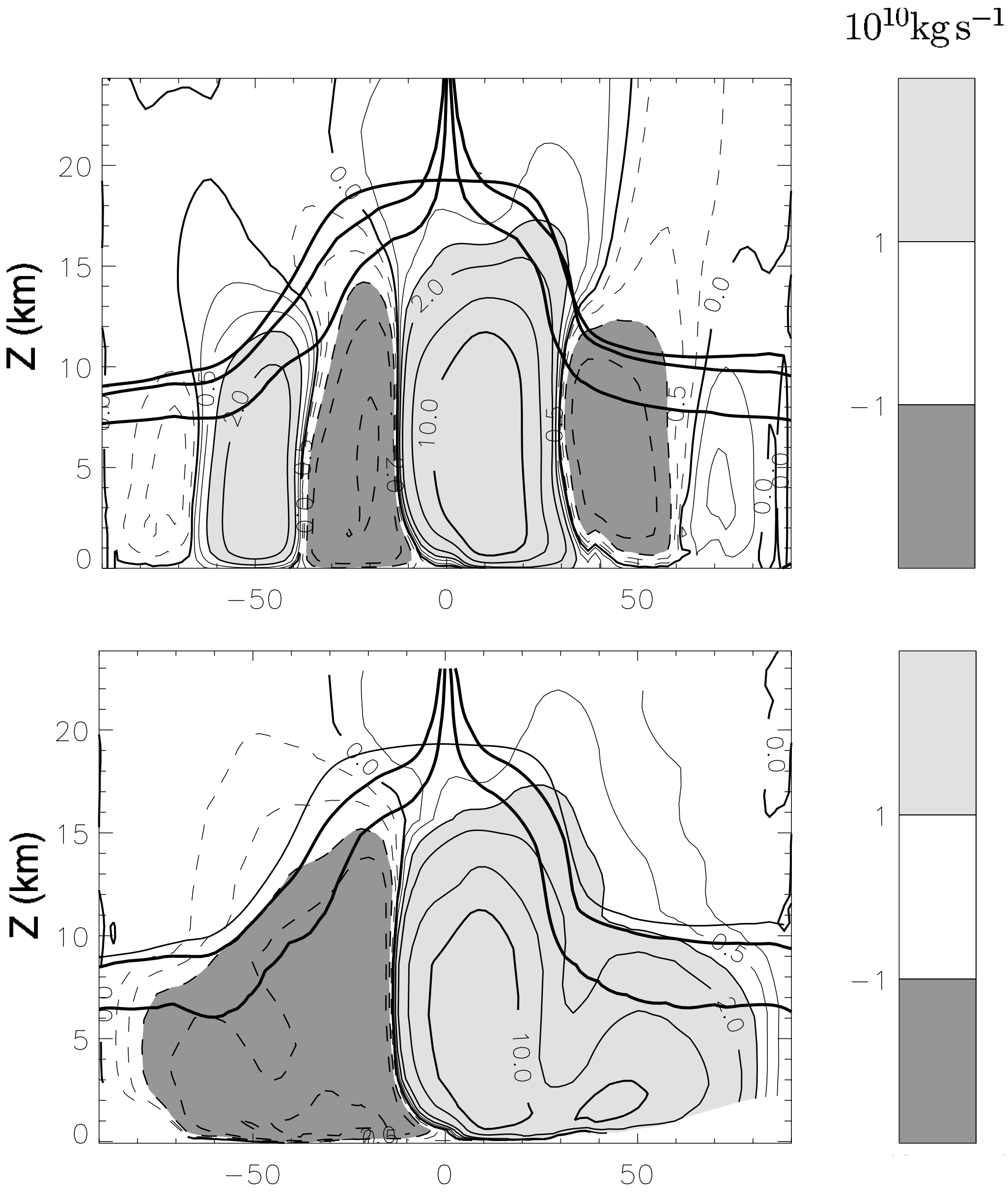


Figure 1.2 Overturning circulation of the atmosphere during a Northern Hemisphere winter. The contours and shading indicate an overturning streamfunction, rising just south of the equator. The top plot shows a conventional Eulerian average and the bottom plot is a residual circulation. Three measures of the tropopause are indicated with the more nearly horizontal solid lines.

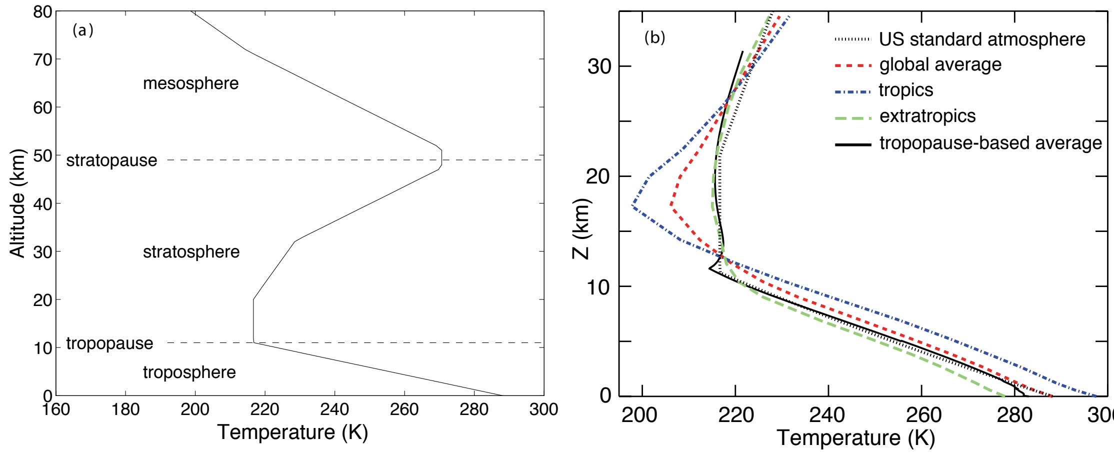


Figure 1.3 Temperature profiles in the atmosphere. On the left is the 'US standard atmosphere' and on the right are some observed profiles.

the above expressions over wavelength or frequency, respectively, gives the Stefan–Boltzmann law

$$B(T) = \sigma T^4, \tag{1.2}$$

where σ is Stefan's constant,

$$\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2} = 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}. \tag{1.3}$$

The maximum of Planck's function occurs at a wavelength $\lambda_m = b/T$ where $b = 2.898 \times 10^{-3}$ m K. This is Wien's displacement law, and it means that the higher the temperature the shorter the wavelength at which emission predominantly occurs. For the Sun, at $T \approx 6000$ K, $\lambda_m = 5{\times}10^{-7}$ m, which is in the visible range; solar radiation is also sometimes called shortwave. For Earth, at $T = 280$ K, $\lambda_m = 1 \times 10^{-5}$ m, which is in the so-called infra-red, sometimes called longwave. The radiation itself is in units of W m−2, and so is a flux of energy. The radiation reaching Earth from the Sun has an intensity of $S^* = 1366\,\mathrm{W\,m^{-2}}$, varying by about 1 W m^{-2} over the 11-year sunspot cycle.

1.2.2 Earth's global energy budget

The simplest model that gives the temperature of the Earth is to suppose that the incoming solar radiation is balanced by an outgoing flux of infra-red radiation at a single temperature so that

$$S_0(1 - \alpha) = \sigma T^4, \tag{1.4}$$

where $S_0 = S^*/4 = 342\,\mathrm{W\,m^{-2}}$ and α is the Earth's albedo, the fraction of solar radiation reflected and measurements show that $\alpha \approx 0.3$. The resulting temperature, T_e is variously called

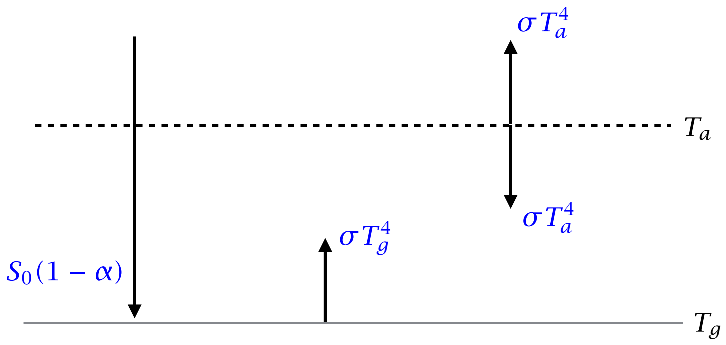


Figure 1.4 A simple EBM

the effective emitting temperature, the radiation temperature or the bolometric temperature. Plugging in numbers we find

$$T_e = \left(\frac{342 \times 0.7}{5.67 \times 10^{-8}}\right)^{1/4} = 255\,\text{K}. \tag{1.5}$$

The actual surface temperature on Earth averages 288 K. If you think 255 K is a good estimate of 288 K, you are at heart a planetary scientist. If you think it is a bad estimate, you are a climate scientist or a meteorologist.

[Needed: table of emitting temperature and actual surface temperature for all planetary bodies in the solar system.]

A simple feedback we can put into such a model is the ice-albedo feedback, whereby we suppose that α is a function of temperature. For example, we might suppose that

$$\alpha = \begin{cases} 0.3 & \text{for } T > T_0 \\ 0.8 & \text{for } T < T_0 \end{cases} \tag{1.6}$$

1.2.3 Effects of the atmosphere

The clear-sky atmosphere is largely transparent to solar radiation, but not to infra-red radiation. Most (but not all) of the solar radiation impinging on the atmosphere that is not reflected by clouds is thus absorbed at the Earth's surface, whereas most of the infra-red radiation emitted at the Earth's surface is absorbed by the atmosphere.

Given this, the next simplest model is to suppose there is an absorbing atmosphere above the surface, as illustrated in Fig. 1.4. If it is in equilibrium then the energy balance equations are:

$$\text{Top:} \qquad S_0(1-\alpha) = \sigma T_a^4, \tag{1.7}$$

$$\text{Surface:} \qquad S_0(1-\alpha) + \sigma T_a^4 = \sigma T_g^4. \tag{1.8}$$

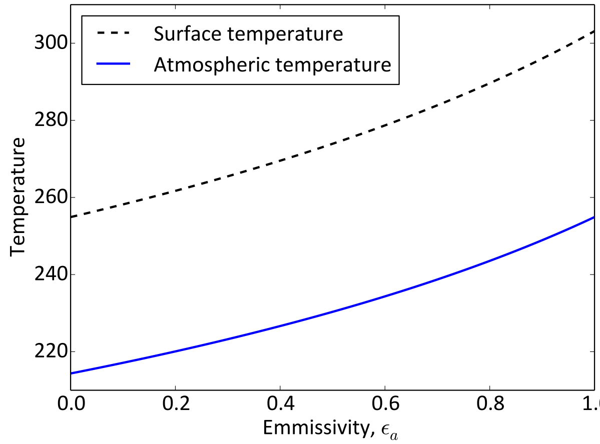


Figure 1.5 Temperature as a function of emissivity in the EBM

(From these we also see the atmospheric balance, $2\sigma T_a^4 = \sigma t_g^4$.) The solution is

$$T_a = \left(\frac{(1-\alpha)S_0}{\sigma}\right)^{1/4}, \qquad T_g = 2^{1/4} T_a. \tag{1.9}$$

So that $T_a = 255\,\text{K}$ (as it has to be) and $T_g = 303\,\text{K}$, This is now too warm. One solution is to suppose the atmosphere has a finite emissivity, ϵ_a (which is less than one). This is getting ad hoc, but it will allow us to illustrate a nice effect. Thus,

$$\text{Top:} \qquad S_0(1-\alpha) = \epsilon_a \sigma T_a^4 + (1-\epsilon_a)\sigma T_g^4, \tag{1.10}$$

$$\text{Surface:} \qquad S_0(1-\alpha) + \epsilon_a \sigma T_a^4 = \sigma T_g^4 + F. \tag{1.11}$$

where we also introduce a flux F from surface to atmosphere. The solution for the surface temperature is

$$\sigma T_g^4 = \frac{S_0(1-\alpha) - F/2}{1 - \epsilon_a/2} \tag{1.12}$$

which, for $F = 0$ and $\epsilon = 0.77$, gives $T_g = 288\,\text{K}$. The surface temperature obviously increases with ϵ_a as expected.

1.3 Water Vapour Feedback

1.3.1 Saturation vapour pressure

The two main greenhouse gases are water vapour and carbon dioxide. Carbon dioxide is well mixed and is not volatile (it does not condense at Earthy temperatures). Its value is determined by geological and anthropogenic processes, and we can suppose its value to be specifiable. Water vapour levels are determined by the relative humidity of the atmosphere and, above all else,

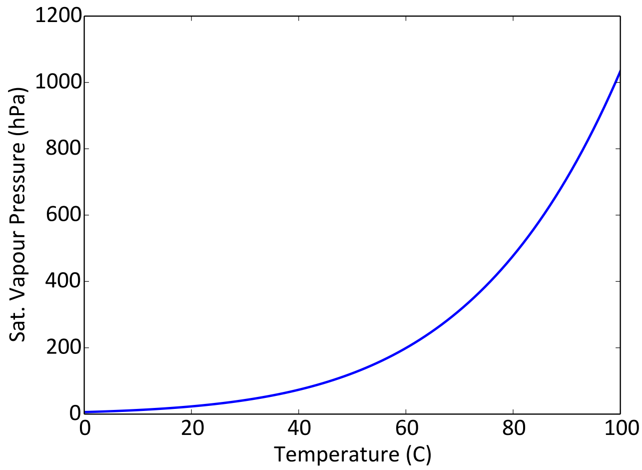


Figure 1.6 Saturation vapour pressure as a function of temperature

by the temperature through the Clausius–Clapeyron relation. This states that the saturation vapour pressure of water, e_s, or indeed of nearly any condensing material, varies as

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L}{T(\rho_g^{-1} - \rho_c^{-1})} \approx \frac{L}{R_w T^2} e_s, \tag{1.13a,b}$$

where the second expression follows if $\rho_g \ll \rho_c$ (the density of the gas phase is much less than that of the condensed phase) and using the ideal gas law. The parameter L is the 'latent heat of condensation' and R_w is the gas constant for the gas in question, which for us is water. If L is constant (not a quantitatively good assumption, but good enough for now) we get

$$e_s(T) = e_{s0} \exp\left[\frac{L_s}{R_s}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]. \tag{1.14}$$

Evidently, saturation vapour pressure is a strongly increasing function of temperature. A liquid will boil when the temperature is sufficiently high that the saturation vapour pressure equals the ambient pressure, and for water at sea-level this occurs at 100° C. A good, semi-empirical approximation for saturation water vapour pressure is the Tetens–Bolton formula,

$$e_s = 6.112 \exp\left(\frac{17.67 * T_c}{T_c + 243.3}\right) \tag{1.15}$$

where T_c is temperature in Celsius and pressure is in hecto-Pascals (the same as millibars). This is actually a better approximation than (1.14) because it includes the variation of L with T. In any case, the main point is that water vapour content in the atmosphere increases fairly rapidly with temperature, at about 7% K^{-1}

1.3.2 Radiative feedback and runaway greenhouse

Returning now to the EBM of the previous section, if we differentiate (1.12) we obtain

$$\frac{4\,\mathrm{d}T_g}{T_g} = \frac{\mathrm{d}\epsilon_a}{2 - \epsilon_a}. \tag{1.16}$$

Now, ϵ_a may vary both because we add CO_2 (which we will denote as c) and because water vapour content may changes, so we write

$$\mathrm{d}\epsilon_a = \mathcal{A}\,\mathrm{d}c + \mathcal{B}\,\mathrm{d}e_s \tag{1.17}$$

where $\mathcal{A}$ and $\mathcal{B}$ are quantities that reflect the radiative properties of CO_2 and water vapour. If the main reason water vapour changes is because of the change in saturation vapour pressure with temperature then, using (1.13b) and (1.17), (1.16) becomes

$$(8 - 4\epsilon_a)\frac{\mathrm{d}T_g}{T_g} = \mathcal{A}\,\mathrm{d}c + \mathcal{B}\,\mathrm{d}e_s = \mathcal{A}, \mathrm{d}c + \frac{\mathcal{B}L}{R_w T^2} e_s\,\mathrm{d}T_g. \tag{1.18}$$

or

$$\left(\frac{8 - 4\epsilon_a}{T_g} - \frac{\mathcal{B}Le_s}{R_w T_g^2}\right)\mathrm{d}T_g = \mathcal{A}\,\mathrm{d}c \tag{1.19}$$

Note that changes in atmospheric temperature is proportional to changes in surface temperature. Thus

$$\frac{\mathrm{d}T_g}{\mathrm{d}c} = \frac{\mathcal{A}T_g}{8 - 4\epsilon_a}\left(\frac{1}{1 - y}\right) \quad \text{where} \quad y = \frac{\mathcal{B}e_s L}{R_w T_g (8 - 4\epsilon_a)}. \tag{1.20}$$

This is a rather interesting equation. It is not to be believed at a quantitative level, but it is perhaps the simplest model that captures in a physically plausible way the greenhouse-gas effects of both water vapour and CO_2. The following is apparent:

- Adding carbon dioxide to the atmosphere causes temperature to go up (because $4\epsilon_a < 8$), providing $y < 1$, so the model delineates between forcing and feedback.
- The feedback is captured by the terms involving y, and it can be larger than the direct effect depending on the size of $\mathcal{B}$.
- As $y \to 1$ the feedback becomes very large, and this is called the *runaway greenhouse effect*. As the temperature increases the water vapour content increases, temperature further increases and so on.
- There is no a priori reason why y should be less than unity. For example, it will be large if the temperature is high, and so if e_s is high. It seems then that that T_g will *decrease* as c increases!

The last item seems totally unphysical, and to see what is going on we need to construct an explicit model of the greenhouse effect with water vapour feedback. We will do that soon but it will be easier if we must look in a bit more detail about radiation.

1.4 Radiative Transfer in a Grey Atmosphere

1.4.1 Assumptions

Radiative intensity, I is the radiative flux per solid angle and when dealing with radiation in three-dimensional problems we have to deal with directionality. We also have to deal with the dependence of absorption on wavelength, and with scattering. In dealing with radiation in the Earth's atmosphere we will make a number of main simplifications.

1. We can have completely separate treatments of solar and infra-red radiation.
2. Much of the time we can assume there is no solar absorption in the atmosphere. This is not quantitatively true but if it were the case, most of the atmosphere would be about the same.
3. We integrate over solid angles in the upward pointing hemisphere and again in the downward pointing atmosphere, so that we have two streams of radiation.
4. We'll integrate over wavelength in the infra-red and assume that a single emissivity suffices.
5. There is no scattering of infra-red radiation.

1.4.2 Equations of radiative transfer

Consider a monochromatic beam of radiation passing through a gas, and suppose for a moment the gas does not emit any radiation but only absorbs it. For a thin layer of gas the change in intensity of the beam is then

$$dI = -I d\tau \tag{1.21}$$

where τ is the *optical depth*. The equation may be regarded as a definition of optical depth — it is the fraction of the incoming radiation absorbed — with the difficulty then arising in relating it to the physical properties of the gas. Eq. (1.21) can be formally integrated to give $I = I_0 \exp(-\tau)$, where the factor $T = \exp(-\tau)$ is the *transmittance* of the layer. The optical depth of two layers is the sum of their optical depths and the total transmittance is the product of the two transmittances. Th

The optical depth of a gas is related both to the amount of gas and to its properties, and for a thin layer of gas of thickness ds we can write

$$d\tau = k_A \rho ds \tag{1.22}$$

where k_A is the mass absorption coefficient. In general the optical depth will depends on the wavelength but we shall assume it does not; that is, the atmosphere is *grey*. In the atmosphere if the pressure is hydrostatic then, in the vertical direction, $d\tau = k_A \rho\, dz = k_A dp$ so that

$$\tau(p_1, p_2) = k_A (p_1 - p_2) \tag{1.23}$$

In fact the mass absorption coefficient increases with pressure so that in the atmosphere a somewhat better approximation is to write

$$\tau \approx \tau_r \frac{(p_1 - p_2)(p_1 + p_2)/2}{p_r^2} \tag{1.24}$$

where p_r is a reference pressure and τ_r is a reference optical depth, a function of the properties of the gas in question.

The slab of gas will also emit radiation so taking this into account (1.21) becomes

$$\mathrm{d}I = (B - I)\mathrm{d}\tau \tag{1.25}$$

This is known as the Schwarzschild equation and it applies at each wavelength, but if we assume τ is not a function of wavelength and we integrate over all wavelengths then $B = \sigma T^4$. (You can either take this to be obvious or do a bit of algebra involving integrations over solid angles to convince yourself, or consult a radiation book like Goody or Petty or Pierrehumbert.) In terrestrial applications we assume that (1.25) applies in the infra-red, and do a separate calculation for solar radiation.

Now, in the atmosphere under two-stream approximation in the atmosphere we have upward, U, and downward, D, radiation and we write

$$-\frac{\mathrm{d}U}{\mathrm{d}\tau} = B - U, \qquad \frac{\mathrm{d}D}{\mathrm{d}\tau} = B - D. \tag{1.26a,b}$$

The *convention* we have chosen here is that τ increases downwards. This is convenient for atmospheric applications, for then we have $\tau = 0$ at the top of the atmosphere, but it is not mandated. We could choose it the other way and flip the signs of the right-hand sides and no physical result depends on this choice, or on the origin of τ. We will use these equations for the infra-red radiation and in what follows assume that solar radiation is all absorbed at the surface.

1.4.3 Solutions

Formal Solution

Consider the generic equation for radiation travelling in the direction of increasing τ or decreasing τ, B and U respectively

$$\frac{\mathrm{d}D}{\mathrm{d}\tau} = B - D, \qquad \frac{\mathrm{d}U}{\mathrm{d}\tau} = U - B. \tag{1.27}$$

Multiplying by the integrating factors $\exp(\tau)$ and $\exp(-\tau)$gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(D\mathrm{e}^{\tau}) = B\mathrm{e}^{\tau}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}(U\mathrm{e}^{-\tau}) = -B\mathrm{e}^{-\tau} \tag{1.28}$$

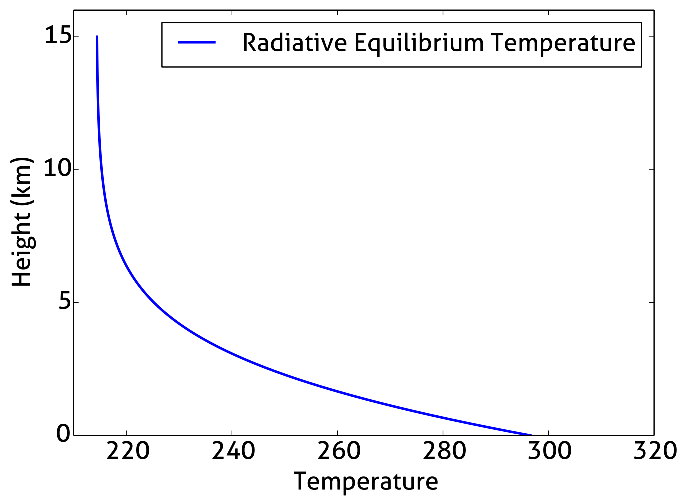


Figure 1.7 Radiative equilibrium temperature (solid curve) calculated using (1.36), with an optical depth of $\tau_0 = 8/3$, $H_a = 2$ km and a net incoming solar radiation of 239 W m^{-2}.

Integrating between $\tau = 0$ and τ' we obtain

$$D(\tau')e^{\tau'} - D(0) = \int_0^{\tau'} B(\tau)e^{\tau}\,d\tau, \qquad U(\tau')e^{-\tau'} - U(0) = -\int_0^{\tau'} B(\tau)e^{-\tau}\,d\tau \tag{1.29}$$

or

$$D(\tau') = e^{-\tau'}\left[D(0) - \int_0^{\tau'} B(\tau)e^{\tau}\,d\tau\right], \qquad U(0) = U(\tau')e^{-\tau'} + \int_0^{\tau'} B(\tau)e^{-\tau}\,d\tau \tag{1.30}$$

The first term in each solution is the attenuation of incoming radiation and the second is the cumulative emission. There are other ways to write the solution, but in general the solution of radiative problems can be written only in the form of integrals. Nevertheless, in some important special cases we can get a local solution as below.

Radiative equilibrium in planetary atmospheres

Consider an atmosphere with net incoming solar radiation S_{net} and suppose the planet is in radiative equilibrium with the incoming solar balanced by outgoing infra-red. The radiative transfer equations are thus to be solved with the boundary conditions that

$$D = 0,\ U = U_t \quad \text{at} \quad \tau = 0, \tag{1.31}$$

where $U_t = S_{net}$ is the net outgoing long-wave radiation (OLR) at the top of the atmosphere. There are still too many variables as we don't know B, but we can obtain a *radiative equilibrium* solution if we assume there is no longwave heating in the column. The heating is proportional to the divergence of the net flux, so that if this is presumed zero then $\partial(U - D)/\partial z = 0$ so that

$$\frac{\partial(U - D)}{\partial\tau} = 0. \tag{1.32}$$

Let us rewrite (1.26) as

$$\frac{\partial}{\partial \tau}(U - D) = U + D - 2B, \tag{1.33a}$$
$$\frac{\partial}{\partial \tau}(U + D) = U - D. \tag{1.33b}$$

A solution of these equations that satisfies the boundary conditions is

$$D = \frac{\tau}{2}U_t, \qquad U = \left(1 + \frac{\tau}{2}\right)U_t, \qquad B = \left(\frac{1+\tau}{2}\right)U_t. \tag{1.34}$$

where U_t is the outgoing longwave radiation at the top of the atmosphere. The only thing remaining is to related τ to z, and a simple recipe that is similar to (1.23) is to suppose that τ has an exponential profile.

$$\tau(z) = \tau_0 \exp(-z/H_a) \tag{1.35}$$

where typical values are $\tau_0 \approx 4$ and $H_a \approx 2\,\mathrm{km}$. The temperature then goes like

$$T^4 = U_t \left(\frac{1 + \tau_0 e^{-z/H_a}}{2\sigma}\right), \tag{1.36}$$

as illustrated in Fig. 1.7. Note the following aspects of the solution.

1. Temperature increases rapidly with height near the ground.

2. The upper atmosphere is nearly isothermal.

3. The temperature at the top of the atmosphere, T_t is given by

$$\sigma T_t^4 = \frac{U_t}{2} \tag{1.37}$$

   Thus, if we define the emitting temperature, T_e, to be such that $\sigma T_e^4 = U_t$, then $T_t = T_e/2^{1/4}$. Note also $B_t/U_t = 1/2$.

In fact, the temperature gradient near the ground varies so rapidly it is likely to be convectively unstable, which we come to in the next chapter. Also, note that we do not need to impose a temperature boundary condition at the ground; in fact there is no ground in this problem! — but what happens if we add one? That is, suppose that we declare that there is a black surface at some height, say $z = 0$, and we require that the atmosphere remain in radiative equilibrium. What temperature does that surface have to be?

From (1.34) the upward irradiance and temperature at any height z are related by

$$U = \left(\frac{2+\tau}{1+\tau}\right)\sigma T^4. \tag{1.38}$$

At $z = 0$ the surface will have to supply upwards radiation equal to that given by (1.38), and therefore its temperature, T_g is given by

$$\sigma T_g^4 = \left(\frac{2+\tau}{1+\tau}\right)\sigma T_s^4, \tag{1.39}$$

where T_s is the temperature of the fluid adjacent to the ground (the 'surface temperature'). That is, $T_g > T_s$ and there is a temperature discontinuity at the ground. Sometimes in very still conditions a very rapid change of temperature near the ground can in fact be observed, but usually the presence of conduction and convection will ensure that T_g and T_s are equal.

In the limit in which $\tau = 0$ in the upper atmosphere (let us prematurely call this the 'stratosphere') then we see that

$$D = 0, \qquad U = U_t, \qquad B = \frac{U_t}{2}. \tag{1.40}$$

That is, the atmosphere is isothermal, there is no downwelling irradiance and the upward flux is constant. The stratospheric temperature, T_{st} and the emitting temperature are related by

$$T_{st} = \frac{T_e}{2^{1/4}}. \tag{1.41}$$

Summary Points

To sum up, what have we found?

1. If we suppose the atmosphere is gray, and we know how optical depth varies with height, then if the atmosphere is in longwave radiative equilibrium we can construct and explicit solution for the temperature as a function of height.

2. The temperature will typically decrease very rapidly in height away from the surface. So much so it is likely to be convectively unstable, as we discuss in the next chapter.

3. The radiative equilibrium temperature does not care or know whether a surface (i.e., the ground) is present. If a surface *is* present, and we require that radiate equilibrium still hold, the temperature of the ground must be higher than the temperature of the air adjacent to it. This is because the ground must supply the same amount of radiation as would be supplied by an infinite layer of air below that level. Thus, there is a temperature discontinuity at the ground, which in reality would normally be wiped out by convection.

1.5 An explicit model of the Runaway Greenhouse Effect

We now come back to the greenhouse effect and construct an explicit model of runaway greenhouse. (The term 'runaway greenhouse' was coined by Ingersoll 1969.) Suppose the atmosphere is in radiative equilibrium. From the derivations above can relate the surface and ground temperatures to the incoming solar radiation through the relation

$$T_s^4 = \frac{T_e^4}{2}(1+\tau_0), \qquad T_g^4 = T_e^4(1+\frac{\tau_0}{2}) \tag{1.42}$$

Thus, if $\tau_0 = 1.254$ then, for $T_e = 255\,\mathrm{K}$ we find $T_g = 288\,\mathrm{K}$ and $T_s = 262\,\mathrm{K}$. The assumption of radiative equilibrium and the ensuing temperature discontinuity are unrealistic but the model will illustrate and important point. We'll construct a more realistic model in the next chapter.

Suppose that we let τ_0 be a function of temperature, increasing with the saturation vapor pressure at the surface. Thus, let

$$\tau_0 = A + Be_s(T_g) \tag{1.43}$$

where A and B are semi-empirical constants, and e_s is the saturation vapor pressure as given by the solution of the Clausius–Clapeyron equation, (1.15). We will tune their values such that $T_g = 288$ when $T_e = 255$, and with some experience of hindsight we set the ratio $A/B = 8$, whence we obtain $A = 1.12$ and $B = 0.14$. The reason for such a seemingly high ratio is that a gray model is too prone to give a runaway greenhouse because of its lack of windows in the infra-red. Thus, in reality, even as temperature and water vapor content increase some infra-red radiation *can* escape from the surface.

Putting the above together, the ground temperature is solution of

$$T_g^4 = T_e^4\left(1 + \frac{1}{2}[A + Be_s(T_g)]\right). \tag{1.44}$$

This algebraic equation is quite nonlinear and must be solved numerically but a few points are apparent.

1. For any given T_e we can obtain a graphic solution by plotting T_g and $T_e^4(1 + \tau_0/2)^{1/4}$ and seeing where the two curves intersect. For a range of values of T_e we will obtain two solutions, as illustrated in Fig. 1.8. However, if T_e is too high there will be no intersection of the curves because the value of $T_e^4(1 + \tau_0/2)$ will always be larger than T_g.
2. If T_e increases and Be_s is much smaller than A, then a solution is found by increasing T_g.
3. If T_e increases and Be_s is suitably large then we can imagine that a solution will be found with a *lower* value of T_g.

Numerical solutions are found iteratively are illustrated in Fig. 1.9, and as expected there are two branches to the solution. [A much more detailed discussion with many extensions is to be found in the report by P. Martin in this volume.] For the parameters plotted, there is no solution if $T_e > 269\,\mathrm{K}$. That is to say, if a planet obeying the model above were in an orbit such that $T_e > 269\,\mathrm{K}$ then infra-red radiation would not be able to escape from the surface, and the surface temperature would keep on rising. All the water on the planet surface would boil, and eventually the water vapor would escape to space. Such a scenario may have occurred on Venus in the past.

1.5.1 Stability of solutions

The upper branch of the solution plotted in Fig. 1.9 runs counter to our intuition, in that temperature decreases as emitting temperature increases. The situation arises because the greenhouse effect is so strong, so that an increase in emitting temperature can lead to a decrease in

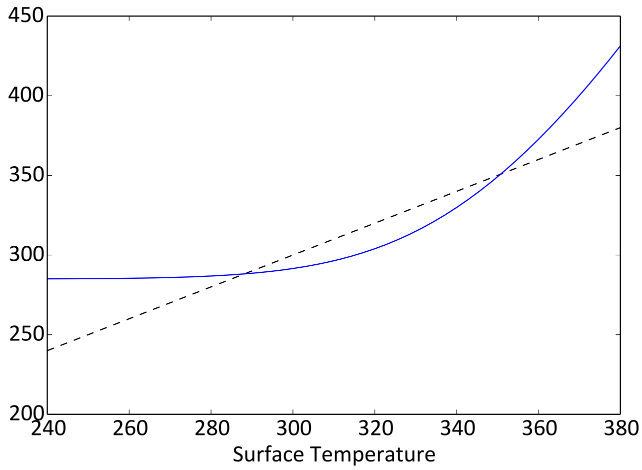


Figure 1.8 Graphical solution to the energy balance model (1.44) with $T_e = 255\,\mathrm{K}$. The dashed curve is T_g and the solid curve plots values of $T_e^4(1 + \tau_0(T_g)/2)^{1/4}$, with τ_0 given by (1.43). Solutions occur at $T_g \approx 288\,\mathrm{K}$ and $T_g = 350\,\mathrm{K}$.

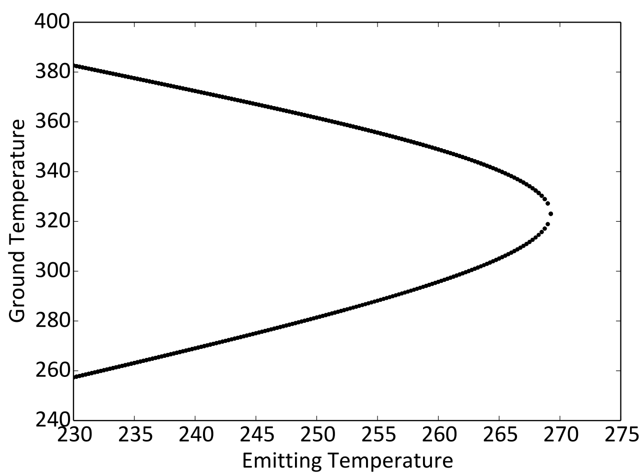


Figure 1.9 Solutions the energy balance model (1.44) obtained numerically. Plotted are values of T_g as a function of The dashed curve is T_g and the solid curve plots values of $T_e^4(1 + \tau_0(T_g)/2)^{1/4}$, with τ_0 given by (1.43) .

surface temperature if the greenhouse effect also falls considerably. However, this solution is *unstable* as we now show.

We add a time dependence to the energy balance model and write

$$C\frac{\mathrm{d}T_g}{\mathrm{d}t} = \sigma T_e^4 - \sigma\frac{T_g^4}{1 + \tau_0/2} \tag{1.45}$$

We perturb the system about an equilibrium point and so obtain

$$C\frac{\mathrm{d}T_g'}{\mathrm{d}t} = \frac{-4\sigma T_g^3 T_g'}{1 + \tau_0} + \frac{\sigma T_g^4 \tau_0'}{(1 + \tau_0)^2}, \tag{1.46a}$$

$$= \left(\frac{T_g}{1 + \tau_0/2} \frac{\mathrm{d}\tau_0}{\mathrm{d}T_g} - 4 \right) T_g' \tag{1.46b}$$

Thus, the solution will be stable or unstable according as whether the term in brackets is negative or positive, respectively.

A tiny bit of algebra will reveal that the ratio of the two terms in brackets in (**??**) precisely the same as the ratio of the gradients of the solid curve and the dashed curve at the intersection points in Fig. 1.8. Thus, the solution at the higher temperature (about 350 K in the graph) is unstable, because the gradient of the blue curve is greater than the gradient of the dashed curve. Similarly, the solution at the lower temperature (288 K) is stable. All of the solutions on the upper branch on Fig. 1.9 are therefore unstable.

Chapter 2

Radiative-Convective Equilibrium and the Height of the Tropopause

We now consider what the effect of convection might be on all the concepts and solutions found in chapter 1. Because our interest in mainly in the large scale structure of the atmosphere we will take a somewhat simplistic view of convection and suppose that it acts to restore an unstable lapse to something that is neutrally stable, that lapse rate being given by either the dry adiabatic or moist adiabatic lapse rate. Readers interested in finding out more about convection and radiative-convective equilibrium should consult Kerry Emanuel's lecture notes.

2.1 Radiative-convective equilibrium

In chapter one we found that in radiative equilibrium the temperature falls off very rapidly with height in the lower atmosphere, so much so that it is likely to be convectively unstable. We imagine the atmosphere will convect and that the lapse rate will adjust until it is stable, as in Fig. 2.1, up to some height H_T. Sometimes, either instead of or in addition to, heat may be transported upwards by large-scale motion such as baroclinic waves. In either case, let us suppose that the dynamics acts such as to produced constant lapse rate up to some height H_T, which we will later associate with the tropopause. We wish to obtain an expression for that height. That is, we seek a solution for which

$$z \le H_T : \qquad T = T_s - \Gamma z \tag{2.1a}$$

$$z > H_T : \qquad \text{Radiative equilibrium, satisfying (1.26) and (1.32)} \tag{2.1b}$$

Further, since we are imposing a convective heat flux, we can suppose that at the surface the temperature is continuous, so that the ground temperature is such that $\sigma T_g^4 = U(z = 0)$.

To obtain a solution we might just think of adjusting the lapse rate in Fig. 1.7 so that there is no net heating, and this may indeed be what convection does on a short timescale. However, an overall radiative balance is not necessarily then achieved, so that the system will then evolve

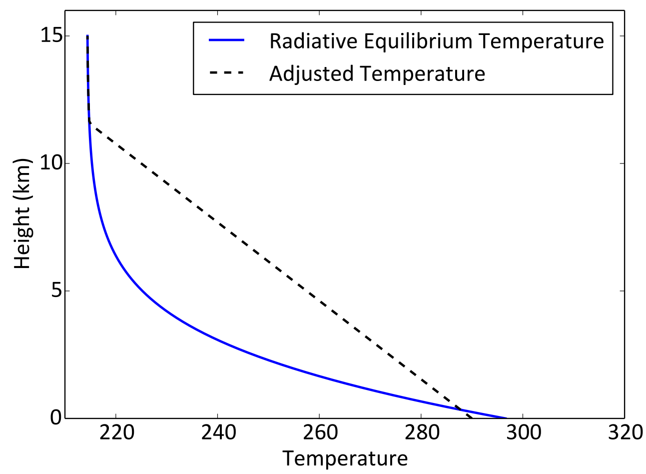


Figure 2.1 Radiative equilibrium temperature (solid curve) calculated using (1.36), with an optical depth of $\tau_0 = 8/3$, $H_a = 2\,\text{km}$ and a net incoming solar radiation of $239\,\text{W m}^{-2}$. The dashed line shows a schematic adjusted temperature with a lapse rate of $6.5\,\text{K km}^{-1}$ up to a tropopause (at about 11 km here) and a radiative equilibrium temperature in the stratosphere.

further. The variable in this equation is H_T, and this can be adjusted until (2.1) is satisfied, with the outgoing radiation The solution of these equations requires an iterative approach and the algorithm is as follows.

1. First solve the radiative transfer equations for radiative equilibrium.
2. Make a guess for the height of the tropopause, and hence obtain the temperature all the way down to the ground.
3. Integrate the radiative transfer equations down from the top. The outgoing radiative balance is achieved this way, but there is no balance at the surface if temperature is continuous. That is, $\sigma T_g^4 \neq U(z = 0)$.
4. Change the height of the tropopause, find another solution, and iterate until the surface radiative balance is achieved.

An alternative is to specify the surface temperature and integrate the radiative transfer equations up along a given lapse rate from the bottom to a certain height, beyond which we suppose that radiative equilibrium holds. This procedure will not give the correct outgoing radiation, so the procedure must again be iterated.

2.1.1 Global Warming

Without actually solving the RCE equations we can make an important deduction as to what happens to the height of the tropopause under global warming, that is what happens when additional carbon dioxide is added to the atmosphere. If the atmosphere stays in radiative balance (which it will in the long term) then the outgoing radiation remains the same. If the stratosphere has a small optical depth then its temperature stays the same from (1.40). Therefore *the temperature of the tropopause must stay the same!* However, the height of the emitting temperature must increase, because the emissivity of the lower atmosphere increases, and the

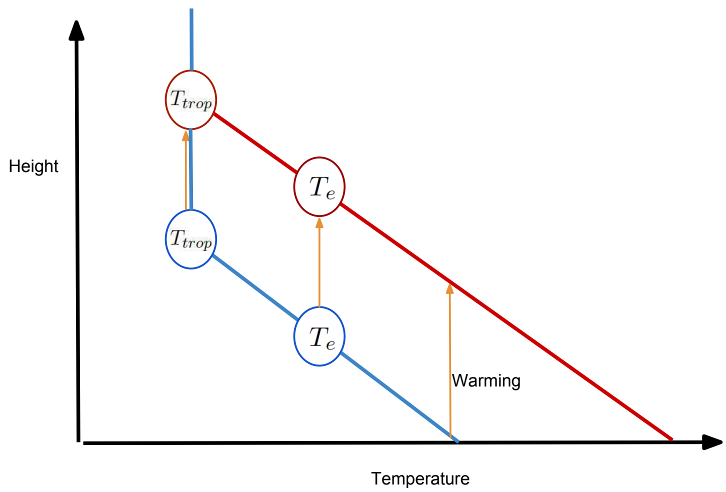


Figure 2.2 Schematic of temperatures before (blue line) and after an increase in optical depth of the atmosphere, such as happens in global warming. The troposphere warms but the emitting temperature stays the same. Hence the tropopause temperature stays the same and the height of the tropopause increases.

photons that reach space come, on average, from a higher level in the troposphere. And, as a consequence, the troposphere warms as illustrated in Fig. 2.2. But if the temperature of the tropopause is to stay the same then its height must increase, and a simple calculation tells us by how much.

If the lapse rate stays the same then the tropopause height will increase by an amount ΔH_T given by

$$\Delta H_t = \frac{\Delta T_s}{\Gamma} \tag{2.2}$$

where ΔT_s is the change in surface temperature. If we allow the lapse rate to change also, then

$$\Delta H_t = \frac{\Delta T_s}{\Gamma} - H_t \frac{\Delta \Gamma}{\Gamma} \tag{2.3}$$

or

$$\frac{\partial H_t}{\partial T_s} = \frac{1}{\Gamma} - \frac{H_T}{\Gamma}\frac{\partial \Gamma}{\partial T_s}. \tag{2.4}$$

If we suppose that Γ is the moist adiabatic lapse rate then we can calculate this expression analytically, and some results are shown in Fig. 2.3, where the lapse rate is assumed constant with height and a function of surface temperature. It is interesting that the increase in tropopause height is quite significant – about 400 m per degree – and that both the direct temperature effect and the lapse rate effect are important (at least in regions where the lapse rate is moist adiabatic). An increase in tropospheric height is one of the most robust results we have concerning changes of the structure of the atmosphere under global warming, as discussed more in Vallis *et al.* (2014).

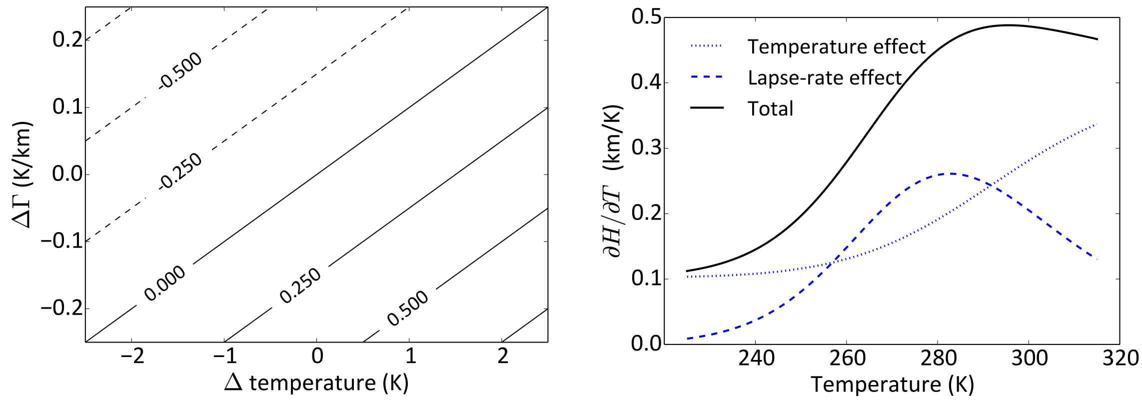


Figure 2.3 (a) Contours of change in tropopause height (km) as a function of temperature change and lapse rate change, calculated using (2.3). (b) Rate of change of tropopause height with temperature ($\partial H_t/\partial T$) as a function of temperature, calculated using (2.4).

2.2 The Height of the Tropopause

We now provide an approximate, analytic, expression for the height of the tropopause.[1] We assume the following.

1. Single column (so a one-dimensional calculation).
2. Grey atmosphere with an optical thickness that decays exponentially with height.
3. A specified lapse rate to some height H_T, beyond which there is radiative equilibrium.
4. An optically thin atmosphere in the upper troposphere and stratosphere.
5. An overall radiative balance. So the outgoing IR radiation is specified (equal to net incoming solar).
6. No surface temperature discontinuity. So ground temperature equals surface air temperature ($T_g = T_s$), and the upwards radiation at $z = 0$ is given by σT_g^4.

2.2.1 Algorithm

To find an exact solution the equations must be iterated, and an algorithm for that is as follows.

(1) First numerically integrate (1.33) to obtain a radiative equilibrium solution.

(2) Guess a height for the tropopause and thus obtain a temperature at all levels below that, including the ground, using the given lapse rate.

(3) Calculate the radiative fluxes by integration of (1.33) down from the top. The upwards radiation at the ground will in general not equal σT_g^4.

[1]This section is joint work with Pablo Zurita-Gotor.

(4) Adjust the height of the tropopause and repeat step ((2)) and ((3)).

(5) Iterate the calculation until a surface balance is achieved.

An alternative procedure is to guess a surface temperature and integrate the equations up, assuming a constant lapse rate up to a height H_T, with radiative equilibrium beyond. When this is done the temperature at H_T will not be the correct one, and outgoing radiation will not equal to the incoming radiation, and again we have to iterate.

2.2.2 Analytic approximation

The analytic approach involves obtaining an analytic expression for the outgoing radiation for a given temperature profile along the lines of (1.30). The OLR so obtained will be a function of the height of the tropopause, and by making the expression equal to the incoming solar radiation we obtain an expression for the tropopause height. Instead of actually using (1.30) it is easier to solve the equations approximately *ab initio*. We make one other approximation, that the value of B/U varies linearly from the tropopause (where its value is 0.5) to its value at the surface (where $B/U = 1$). Thus,

$$\frac{B}{U} = 1 - \frac{z}{2H_T}. \tag{2.5}$$

Numerical calculations suggest this is a decent approximation (can it be improved upon?). Rewrite (1.26a) as

$$\frac{\mathrm{d}\log U}{\mathrm{d}\tau} = 1 - \frac{B}{U} = \frac{z}{2H_T}. \tag{2.6}$$

Using $\tau(z) = \tau_s \exp(-z/H_a)$ we obtain

$$\frac{\mathrm{d}\log U}{\mathrm{d}z} = -\frac{z}{2H_T H_a}\tau_s \exp(-z/H_a). \tag{2.7}$$

We can integrate this expression by parts to obtain a value of the upwelling radiation at the tropopause $U(H_T)$, namely

$$\log\left(\frac{U(H_T)}{U(0)}\right) = -\frac{\tau_s}{2H_T}\int_0^{H_T} \exp(-z/H_a)\,\mathrm{d}z \approx -\frac{\tau_s H_a}{2H_T}. \tag{2.8}$$

for $H_T \gg H_a$. This is an expression for the outgoing longwave radiation, and we see that the only variable in the equation is H_T – note that the upwelling radiation at the surface is given by the surface temperature, which is a function of the tropopause temperature, H_T and the lapse rate, Γ.

To obtain a closed form for the tropopause height assume that the stratosphere is optically thin and note that $U(H_T) = U(H_T) = 2\sigma T_T^4$ and $U(0) = \sigma T_g^4 = \sigma T_s^4$. Furthermore, T_T and T_s are related by $T_T = T_s - \Gamma H_T$. The left-hand side of (2.8) then becomes

$$\log\left(\frac{2\sigma T_T^4}{\sigma T_s^4}\right) = \log 2 + 4\log\frac{T_T}{T_s} = \log 2 + 4\log\left(\frac{T_T}{T_T + \Gamma H_T}\right)$$

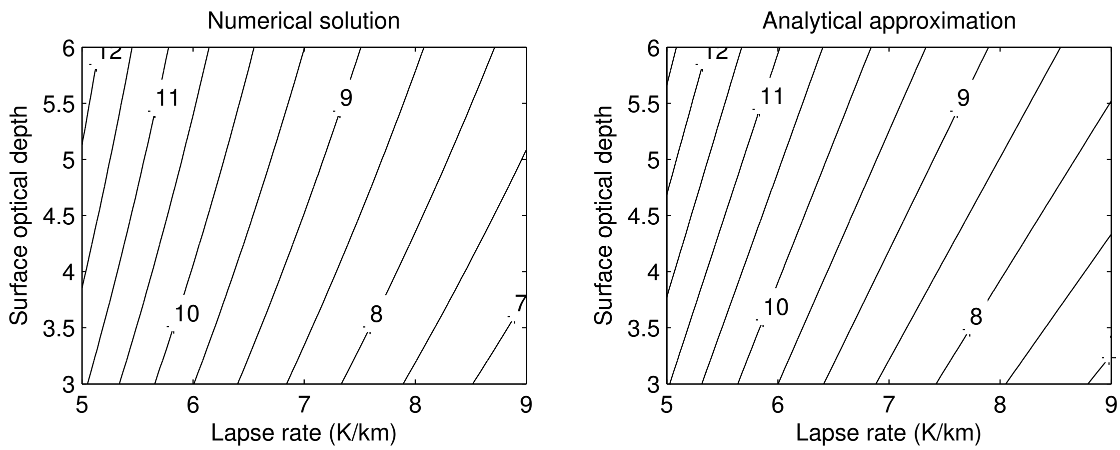


Figure 2.4 Analytic approximation and numerical calculation for tropopause height.

$$\approx \log 2 - \frac{4\Gamma H_T}{T_T}. \tag{2.9}$$

Using (2.9), (2.8) becomes

$$\log 2 - \frac{4\Gamma H_T}{T_s} = -\frac{\tau_s H_a}{2H_T} \tag{2.10}$$

or

$$8\Gamma H_T^2 - \mathrm{C}\, H_T T_T - \tau_s H_a T_T = 0. \tag{2.11}$$

where $\mathrm{C} = 2\log 2 \approx 1.38$. The solution of this equation is

$$H_T = \frac{1}{16\Gamma}\left(\mathrm{C}T_T + \sqrt{\mathrm{C}^2 T_T^2 + 32\Gamma\tau_s H_a T_T}\right) \tag{2.12}$$

For Earth's atmosphere, $H_a \approx 2\,\mathrm{km}$, $\tau_s \approx 8/3$ and $\Gamma \approx 6.5\,\mathrm{K\,km^{-1}}$. All three terms in the quadratic are then approximately the same size and $H_T = 10.3\,\mathrm{km}$, which is in fact reasonably close to the exact numerical solution (obtained iteratively) of the radiative-convective equations (Fig. 2.4).

The numerical approximation of the logarithm in (2.9) can be improved by using T_m instead of T_T, where T_m is the temperature half way between the surface and the tropopause. However we still want to have T_T as a parameter in the quadratic for H_T (because T_T is given if the OLR is known). Thus, we have to do some more algebraic fiddling and the upshot is that we get a quadratic similar to (2.11) but with different coefficients. [Student exercise. See also Vallis et al (2014) for another way to proceed.]

Once we have the tropopause height we can obtain an expression for the temperature everywhere in the troposphere, and the surface. We could then perform a calculation similar to that of section 1.3.2 and obtain an analytic expression for how the surface temperature increases with carbon dioxide content, and the conditions for a runaway greenhouse effect. [With extensions this could be a student project.]

Optically thick and thin limits

The above approach allows us to be precise about what it means for an atmosphere to be optically thin or thick. Using (2.12) and approximating $C^2 = 2$ we easily find that the optically thick limit arises when

$$\tau_s H_a \gg \frac{T_T}{16\Gamma} \quad \text{whence} \quad H_T \approx \sqrt{\frac{T_T \tau_s H_a}{8\Gamma}} \tag{2.13}$$

The optically thin case has

$$\tau_s H_a \ll \frac{T_T}{16\Gamma} \quad \text{whence} \quad H_T \approx \frac{1.38 T_T}{8\Gamma}. \tag{2.14}$$

With parameters appropriate for Earth's atmosphere both of the above limits give estimates in the range 5–10 km, and note that they are additive effects. What is the interpretation of these expressions? Do they work on other planets? What is the role of lateral heat transport?

A number of these issues have been taken up by Shineng Hu in his summer project, and the interested reader is referred to his report for more details.

2.3 Appendix: Dry and Wet Lapse Rates

2.3.1 A dry ideal gas

The negative of the rate of change of the temperature in the vertical is known as the *temperature lapse rate*, or often just the lapse rate, and the lapse rate corresponding to $\partial\theta/\partial z = 0$ is called the *dry adiabatic lapse rate* and denoted Γ_d. Using $\theta = T(p_0/p)^{R/c_p}$ and $\partial p/\partial z = -\rho g$ we find that the lapse rate and the potential temperature lapse rate are related by

$$\frac{\partial T}{\partial z} = \frac{T}{\theta}\frac{\partial \theta}{\partial z} - \frac{g}{c_p}, \tag{2.15}$$

so that the dry adiabatic lapse rate is given by

$$\Gamma_d = \frac{g}{c_p}. \tag{2.16}$$

The conditions for static stability are thus:

$$\begin{aligned} &\text{stability}: \quad \frac{\partial \widetilde{\theta}}{\partial z} > 0; \quad \text{or} \quad -\frac{\partial \widetilde{T}}{\partial z} < \Gamma_d \\ &\text{instability}: \quad \frac{\partial \widetilde{\theta}}{\partial z} < 0; \quad \text{or} \quad -\frac{\partial \widetilde{T}}{\partial z} > \Gamma_d \end{aligned}, \tag{2.17a,b}$$

where a tilde indicates that the values are those of the environment. The atmosphere is in fact generally stable by this criterion: the observed lapse rate, corresponding to an observed buoyancy frequency of about $10^{-2}\,\mathrm{s}^{-1}$, is often about $7\,\mathrm{K\,km}^{-1}$, whereas a dry adiabatic lapse rate is about $10\,\mathrm{K\,km}^{-1}$. Why the discrepancy? One reason, particularly important in the tropics, is that the atmosphere contains water vapour.

2.3.2 Saturated lapse rate

The amount of water vapour that can be contained in a given volume is an increasing function of temperature (with the presence or otherwise of dry air in that volume being largely irrelevant). Thus, if a parcel of water vapour is cooled, it will eventually become saturated and water vapour will condense into liquid water. A measure of the amount of water vapour in a unit volume is its partial pressure, and the partial pressure of water vapour at saturation, e_s, is given by the Clausius–Clapeyron equation,

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L_c e_s}{R_v T^2}, \tag{2.18}$$

where L_c is the latent heat of condensation or vapourization (per unit mass) and R_v is the gas constant for water vapour. If a parcel rises adiabatically it will cool, and at some height (known as the 'lifting condensation level', a function of its initial temperature and humidity only) the parcel will become saturated and any further ascent will cause the water vapour to condense. The ensuing condensational heating causes the temperature and buoyancy of the parcel to increase; the parcel thus rises further, causing more water vapour to condense, and so on, and the consequence of this is that an environmental profile that is stable if the air is dry may be unstable if saturated. Let us now derive an expression for the lapse rate of a saturated parcel that is ascending adiabatically apart from the affects of condensation.

Let w denote the mass of water vapour per unit mass of dry air, the mixing ratio, and let w_s be the saturation mixing ratio. ($w_s = \alpha e_s/(p - e_s) \approx \alpha_w e_s/p$ where $\alpha_w = 0.622$, the ratio of the mass of a water molecule to one of dry air.) The diabatic heating associated with condensation is then given by

$$Q_{cond} = -L_c \frac{\mathrm{D}w_s}{\mathrm{D}t}, \tag{2.19}$$

so that the thermodynamic equation is

$$c_p \frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}, \tag{2.20}$$

or, in terms of p and and T

$$c_p \frac{\mathrm{D}\ln T}{\mathrm{D}t} - R\frac{\mathrm{D}\ln P}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}. \tag{2.21}$$

If these material derivatives are due to the parcel ascent then

$$\frac{\mathrm{d}\ln T}{\mathrm{d}z} - \frac{R}{c_p}\frac{\mathrm{d}\ln p}{\mathrm{d}z} = -\frac{L_c}{Tc_p}\frac{\mathrm{d}w_s}{\mathrm{d}z}, \tag{2.22}$$

and using the hydrostatic relationship and the fact that w_s is a function of T and p we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}z} + \frac{g}{c_p} = -\frac{L_c}{c_p}\left[\left(\frac{\partial w_s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial w_s}{\partial p}\right)_T \rho g\right]. \tag{2.23}$$

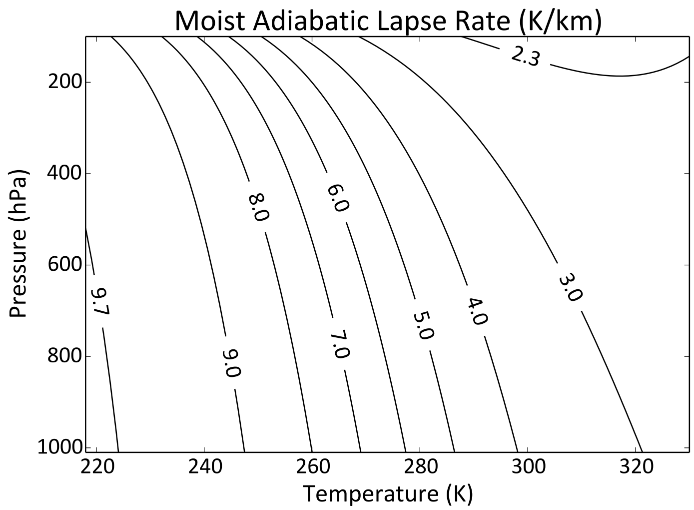


Figure 2.5 The saturated adiabatic lapse rate as a function of temperature and pressure when water (H_2O) is the condensate.

Solving for $\mathrm{d}T/\mathrm{d}z$, the lapse rate, Γ_s, of an ascending saturated parcel is given by

$$\Gamma_s = -\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{g}{c_p}\frac{1-\rho L_c(\partial w_s/\partial p)_T}{1+(L_c/c_p)(\partial w_s/\partial T)_p} \approx \frac{g}{c_p}\frac{1+L_c w_s/(R_d T)}{1+L_c^2 w_s/(c_p R_v T^2)}. \tag{2.24}$$

where the last near equality follows with use of the Clausius–Clapeyron relation. The quantity R_d is the gas constant for dry air and R_v is the gas constant for water vapor, and $R_v = R_d/\alpha_w$. The quantity Γ_s is variously called the *pseudoadiabatic* or *moist adiabatic* or *saturated adiabatic* lapse rate, and it is plotted in Fig. 2.5.

Because g/c_p is the dry adiabatic lapse rate Γ_d, $\Gamma_s < \Gamma_d$, and values of Γ_s are typically around $6\,\mathrm{K\,km^{-1}}$ in the lower atmosphere; however, $\mathrm{d}w_s/\mathrm{d}T$ is an increasing function of T so that Γ_s decreases with increasing temperature and can be as low as $3.5\,\mathrm{K\,km^{-1}}$. For a saturated parcel, the stability conditions analogous to (2.17) are

$$\text{stability}: \qquad -\frac{\partial \widetilde{T}}{\partial z} < \Gamma_s, \tag{2.25a}$$

$$\text{instability}: \qquad -\frac{\partial \widetilde{T}}{\partial z} > \Gamma_s. \tag{2.25b}$$

where $\widetilde{T}$ is the environmental temperature. The observed environmental profile in convecting situations is often a combination of the dry adiabatic and moist adiabatic profiles: an unsaturated parcel that is is unstable by the dry criterion will rise and cool following a dry adiabat, Γ_d, until it becomes saturated at the lifting condensation level, above which it will rise following a saturation adiabat, Γ_s. Such convection will proceed until the atmospheric column is stable and, especially in low latitudes, the lapse rate of the atmosphere is largely determined by such convective processes.

Chapter 3

Rossby Waves and Surface Winds

In this our third lecture we stay with the atmosphere and introduce some dynamics. Our first goal is to understand why there are surface winds, and in particular why there are surface westerlies. A full explanation of this would require a discussion of baroclinic instability and take up a couple of lectures in itself. We'll skip all that and carry out explicit derivations only for the barotropic vorticity equation, with th reader filling in the gaps phenomenologically. We do note that there are westerly winds aloft in the atmosphere because of the thermal wind relation, $f\partial u/\partial z = \partial b/\partial z$, where b is buoyancy which is like temperature. Thus, a temperature gradient between the equator and the pole implies that the zonal wind increases with height. But this doesn't of itself mean that the surface winds are non-zero – we will need momentum fluxes for that. By the same token, momentum fluxes are not needed to have westerly winds aloft.

We begin with a few basic equations.

3.1 Momentum Equation

The zonally-averaged momentum, in Cartesian geometry has the form

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} = \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial \tau}{\partial z} \tag{3.1}$$

where $f = f_0 + \beta y$ In mid-latitudes we usually neglect the mean advection terms ($\overline{\zeta}\overline{v}$ here) which in midlatitudes are small. If we multiply by density and integrate vertically then, in a steady state the terms on the left-hand side both vanish, whence

$$\tau_s = \int_z \rho\overline{u'v'}\,\mathrm{d}z \tag{3.2}$$

where τ_s is the surface stress, which is roughly proportional to the surface wind: $\tau_s \approx r\overline{u}_s$ where r is a constant. Thus

$$\overline{u}_s \approx \frac{1}{r}\int_z \rho\overline{u'v'}\,\mathrm{d}z. \tag{3.3}$$

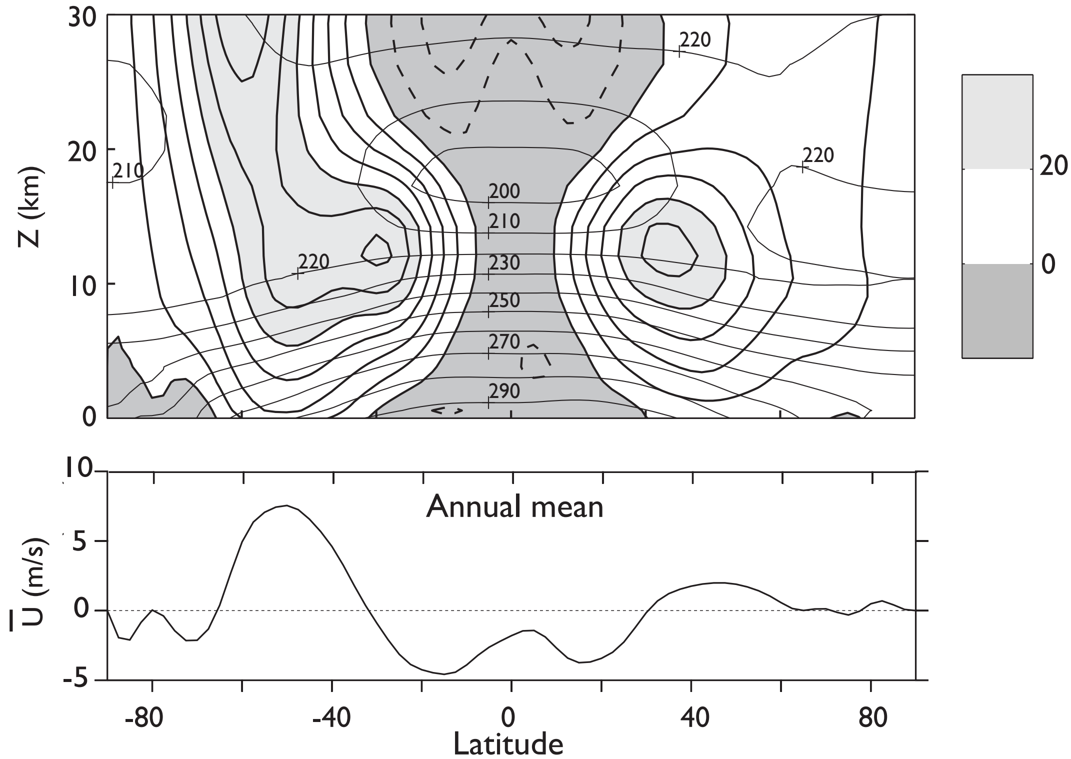


Figure 3.1 (a) Annual mean, zonally averaged zonal wind (heavy contours and shading) and the zonally averaged temperature (lighter contours). (b) Annual mean, zonally averaged zonal winds at the surface. The wind contours are at intervals of 5 m s^{-1} with shading for eastward winds above 20 m s^{-1} and for all westward winds, and the temperature contours are labelled. The ordinate of (a) and (c) is $Z = -H \log(p/p_R)$, where p_R is a constant, with scale height $H = 7.5$ km.

In other words, the surface winds arise because of the eddy convergence of momentum in the atmosphere. Where does this come from? It turns out that it arises from the sphericity of the Earth which gives rise to differential rotation and Rossby waves, as we shall see.

3.2 Rossby Waves: a Brief Tutorial

The inviscid, adiabatic potential vorticity equation is

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \nabla q = 0, \tag{3.4}$$

where $q(x, y, z, t)$ is the potential vorticity and $\boldsymbol{u}(x, y, z, t)$ is the horizontal velocity. The velocity is related to a streamfunction by $u = -\partial\psi/\partial y,\ v = \partial\psi/\partial x$ and the potential vorticity is some function of the streamfunction, which might differ from system to system. Two examples, one applying to a continuously stratified system and the second to a single layer system, are

$$q = f + \zeta + \frac{\partial}{\partial z}\left(S(z)\frac{\partial \psi}{\partial z}\right), \qquad q = \zeta + f - k_d^2\psi. \tag{3.5a,b}$$

We deal mainly the second. If the basic state is a zonal flow and purely a function of y then

$$q = \overline{q}(y,z) + q'(x,y,t), \qquad \psi = \overline{\psi}(y,z) + \psi'(x,y,z,t) \tag{3.6}$$

whence

$$\frac{\partial q'}{\partial t} + \overline{\boldsymbol{u}}\cdot\nabla\overline{q} + \overline{\boldsymbol{u}}\cdot\nabla q' + \boldsymbol{u}'\cdot\nabla\overline{q} + \boldsymbol{u}'\cdot\nabla q' = 0. \tag{3.7}$$

Linearizing gives

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{3.8}$$

3.2.1 Rossby waves in a single layer

In the single-layer case we have $q = \beta y + \nabla^2\psi - k_d^2\psi$. If we linearize this around a zonal flow then $\psi = \overline{\psi} + \psi'$ and

$$\overline{\psi} = -\overline{u}y \qquad \overline{q} = \beta y + \overline{u}k_d^2 y \tag{3.9}$$

and

$$q' = \nabla^2\psi' - k_d^2\psi' \tag{3.10}$$

and (3.8) becomes

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)(\nabla^2\psi' - \psi' k_d^2) + \frac{\partial \psi'}{\partial x}(\beta + Uk_d^2) = 0 \tag{3.11}$$

Substituting $\psi' = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}$ we obtain the dispersion relation,

$$\omega = \frac{k(UK^2 - \beta)}{K^2 + k_d^2} = Uk - k\frac{\beta + Uk_d^2}{K^2 + k_d^2}. \tag{3.12}$$

We will simplify by taking $U = 0$ whence

$$\omega = -\frac{\beta}{K^2 + k_d^2}. \tag{3.13}$$

The corresponding components of phase speed and group velocity are

$$c_p^x \equiv \frac{\omega}{k} = -\frac{\beta}{K^2 + k_d^2}, \qquad c_p^y \equiv \frac{\omega}{l} = \frac{k}{l}\left(\frac{\beta}{K^2 + k_d^2}\right) \tag{3.14a,b}$$

and

$$c_g^x \equiv \frac{\partial \omega}{\partial k} = \frac{\beta(k^2 - l^2 - k_d^2)}{(K^2 + k_d^2)^2}, \qquad c_g^y \equiv \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{(K^2 + k_d^2)^2}, \tag{3.15a,b}$$

which $K^2 = k^2 + l^2$.

3.3 Momentum transport in Rossby waves

It turns out that Rossby waves will transport momentum from place to place, and this is why we have surface winds! (Well, at least it is an explication of why we have surface winds. Other explications that don't involve Rossby waves can be given (Vallis 2006)) but they are all really the same explanation.)

Let us suppose that some mechanism is present that excites Rossby waves in mid-latitudes. This mechanism is in fact baroclinic instability, but we don't really need to know that. We expect that Rossby waves will be generated there, propagate away and break and dissipate. To the extent that the waves are quasi-linear and do not interact, then just away from the source region each wave has the form

$$\psi = \mathrm{Re}\, C e^{\mathrm{i}(kx+ly-\omega t)} = \mathrm{Re}\, C e^{\mathrm{i}(kx+ly-kct)}, \tag{3.16}$$

where C is a constant, with dispersion relation

$$\omega = ck = \overline{u}k - \frac{\beta k}{k^2 + l^2} \equiv \omega_R, \tag{3.17}$$

taking $k_d = 0$ and provided that there is no meridional shear in the zonal flow. The meridional component of the group velocity is given by

$$c_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{(k^2 + l^2)^2}. \tag{3.18}$$

Now, the direction of the group velocity must be *away* from the source region; this is a radiation condition, demanded by the requirement that Rossby waves transport energy *away* from the disturbance. Thus, northwards of the source kl is positive and southwards of the source kl is negative. That the product kl can be positive or negative arises because for each k there are two possible values of l that satisfy the dispersion relation (3.17), namely

$$l = \pm \left(\frac{\beta}{\overline{u} - c} - k^2 \right)^{1/2}, \tag{3.19}$$

assuming that the quantity in parentheses is positive.

The velocity variations associated with the Rossby waves are

$$u' = -\mathrm{Re}\, C\, \mathrm{i} l e^{\mathrm{i}(kx+ly-\omega t)}, \qquad v' = \mathrm{Re}\, C\, \mathrm{i} k e^{\mathrm{i}(kx+ly-\omega t)}, \tag{3.20a,b}$$

and the associated momentum flux is

$$\overline{u'v'} = -\frac{1}{2} C^2 kl. \tag{3.21}$$

Thus, given that the sign of kl is determined by the group velocity, northwards of the source the momentum flux associated with the Rossby waves is southward (i.e., $\overline{u'v'}$ is negative), and

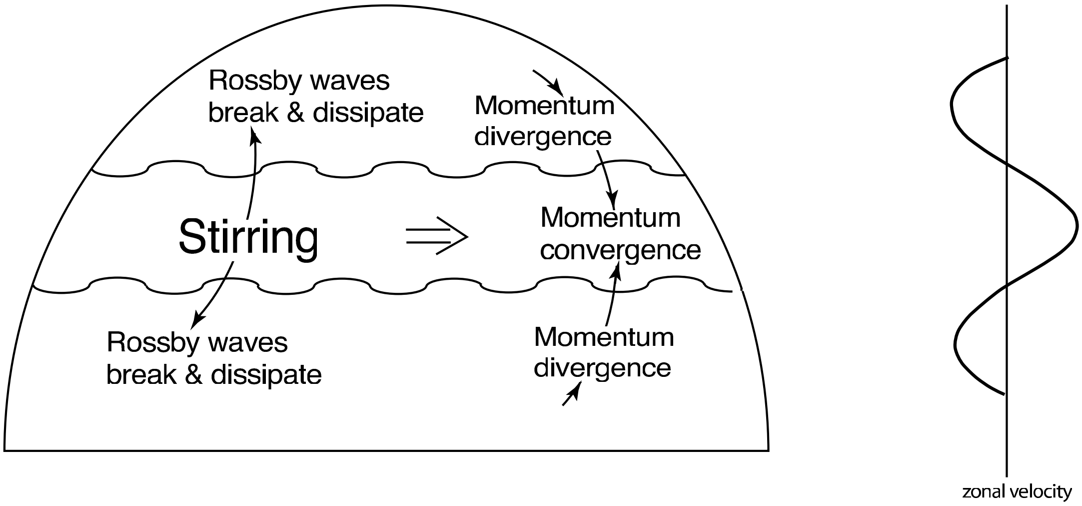


Figure 3.2 Generation of zonal flow on a β-plane or on a rotating sphere. Stirring in mid-latitudes (by baroclinic eddies) generates Rossby waves that propagate away from the disturbance. Momentum converges in the region of stirring, producing eastward flow there and weaker westward flow on its flanks.

southwards of the source the momentum flux is northward (i.e., $\overline{u'v'}$ is positive). That is, the momentum flux associated with the Rossby waves is *toward* the source region. Momentum converges in the region of the stirring, producing net eastward flow there and westward flow to either side (Fig. 3.2).

Another way of describing the same effect is to note that if kl is positive then lines of constant phase ($kx+ly$ = constant) are tilted north-west/south-east, as in Fig. 3.3 and the momentum flux associated with such a disturbance is negative ($\overline{u'v'} < 0$). Similarly, if kl is negative then the constant-phase lines are tilted north-east/south-west and the associated momentum flux is positive ($\overline{u'v'} > 0$). The net result is a convergence of momentum flux into the source region. In physical space this is reflected by having eddies that are shaped like a boomerang, as in Fig. 3.3.

Pseudomomentum and wave–mean-flow interaction

The kinematic relation between vorticity flux and momentum flux for non-divergent two-dimensional flow is

$$v\zeta = \frac{1}{2}\frac{\partial}{\partial x}\left(v^2 - u^2\right) - \frac{\partial}{\partial y}(uv). \tag{3.22}$$

After zonal averaging this gives

$$\overline{v'\zeta'} = -\frac{\partial \overline{u'v'}}{\partial y}, \tag{3.23}$$

noting that $\overline{v} = 0$ for two-dimensional incompressible (or geostrophic) flow.

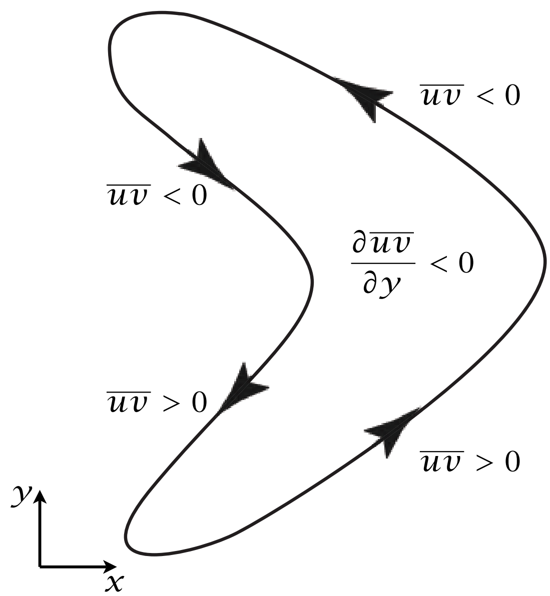


Figure 3.3 The momentum transport in physical space, caused by the propagation of Rossby waves away from a source in mid-latitudes. The ensuing boomerang-shaped eddies are responsible for a convergence of momentum, as indicated in the idealization pictured.

Now, the barotropic zonal momentum equation is (for horizontally non-divergent flow)

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} - fv = -\frac{\partial \phi}{\partial x} + F_u - D_u, \tag{3.24}$$

where F_u and D_u represent the effects of any forcing and dissipation. Zonal averaging, with $\overline{v} = 0$, gives

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial \overline{u'v'}}{\partial y} + \overline{F}_u - \overline{D}_u, \tag{3.25}$$

or, using (3.23),

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} + \overline{F}_u - \overline{D}_u. \tag{3.26}$$

Thus, the zonally averaged wind is maintained by the zonally averaged vorticity flux. On average there is little if any direct forcing of horizontal momentum and we may set $\overline{F}_u = 0$, and if the dissipation is parameterized by a linear drag (3.26) becomes

$$\boxed{\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} - r\overline{u}}\,, \tag{3.27}$$

where the constant r is an inverse frictional time scale.

Now consider the maintenance of this vorticity flux. The barotropic vorticity equation is

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla\zeta + v\beta = F_\zeta - D_\zeta, \tag{3.28}$$

where F_ζ and D_ζ are forcing and dissipation of vorticity. Linearize about a mean zonal flow to give

$$\frac{\partial \zeta'}{\partial t} + \overline{u}\frac{\partial \zeta'}{\partial x} + \gamma v' = F'_\zeta - D'_\zeta, \tag{3.29}$$

where

$$\gamma = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} \tag{3.30}$$

is the meridional gradient of absolute vorticity. Multiply (3.29) by ζ'/γ and zonally average, assuming that $\overline{u}_{yy}$ is small compared to β or varies only slowly, to form the pseudomomentum equation,

$$\frac{\partial \mathcal{A}}{\partial t} + \overline{v'\zeta'} = \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}), \tag{3.31a}$$

$$\mathcal{A} = \frac{1}{2\gamma}\overline{\zeta'^2} \tag{3.31b}$$

is a wave activity density, equal to the (negative of) the pseudomomentum for this problem. The parameter γ is positive if the average absolute vorticity increases monotonically northwards, and this is usually the case in both Northern and Southern Hemispheres.

3.3.1 An Aside on Wave Activity and Stability

Suppose the flow is unforced and inviscid (common conditions that we impose in stability problems). Then the wave activity equation above becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \overline{v'\zeta'} = 0. \tag{3.32}$$

This condition holds even in the presence of shear. Integrating between quiescent latitudes gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathcal{A}\,\mathrm{d}y = 0. \tag{3.33}$$

The quantity $\widehat{A} \equiv \int \mathcal{A}\,\mathrm{d}y$ is wave activity, something that is quadratic in wave amplitude and is conserved. $\mathcal{A}$ itself is a wave activity density. Energy is not normally a wave activity, because it grows if the flow us unstable, whereas a wave activity does not.

Now suppose that γ is positive everywhere. In this case the conservation of $\widehat{A}$ prevents $\overline{\zeta'^2}$ from growing! Thus, for a wave to grow, $\beta - \overline{u}_{yy}$ must change sign somewhere in the domain. We have derived the *Rayleigh-Kuo* criterion for barotropic instability. Note that there is no mention of normal modes, although we have still (in this derivation) assumed linearity.

3.4 Wave–mean-flow interaction, acceleration and non-acceleration

In the absence of forcing and dissipation, (3.27) and (3.31a) imply an important relationship between the change of the mean flow and the pseudomomentum, namely

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \mathcal{A}}{\partial t} = 0. \tag{3.34}$$

We have now essentially derived a special case of the *non-acceleration* result. If the waves are steady and inviscid, then from (3.31a) $\overline{v'\zeta'} = 0$. Then from (3.34) the mean flow does not accelerate. We need to do a bit more work in the stratified case, but the essence of the result is the same.

Now if for some reason $\mathcal{A}$ increases, perhaps because a wave enters an initially quiescent region because of stirring elsewhere, then mean flow must decrease. However, because the vorticity flux integrates to zero, the zonal flow cannot decrease everywhere. Thus, if the zonal flow decreases in regions away from the stirring, it must *increase* in the region of the stirring. In the presence of forcing and dissipation this mechanism can lead to the production of a statistically steady jet in the region of the forcing, since (3.27) and (3.31a) combine to give

$$\frac{\partial \overline{u}}{\partial t} + \frac{\partial \mathcal{A}}{\partial t} = -r\overline{u} + \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}), \tag{3.35}$$

and in a statistically steady state

$$r\overline{u} = \frac{1}{\gamma}(\overline{\zeta' F'_\zeta} - \overline{\zeta' D'_\zeta}). \tag{3.36}$$

The terms on the right-hand side represent the stirring and dissipation of vorticity, and integrated over latitude their sum will vanish, or otherwise the pseudomomentum budget cannot be in a steady state. However, let us suppose that forcing is confined to mid-latitudes. In the forcing region, the first term on the right-hand side of (3.36) will be larger than the second, and an eastward mean flow will be generated. Away from the direct influence of the forcing, the dissipation term will dominate and westward mean flows will be generated, as sketched in Fig. 3.4. Thus, *on a β-plane or on the surface of a rotating sphere an eastward mean zonal flow can be maintained by a vorticity stirring that imparts no net momentum to the fluid.* In general, stirring in the presence of a vorticity gradient will give rise to a mean flow, and on a spherical planet the vorticity gradient is provided by differential rotation.

It is crucial to the generation of a mean flow that the dissipation has a broader latitudinal distribution than the forcing: if all the dissipation occurred in the region of the forcing then from (3.36) no mean flow would be generated. However, Rossby waves are generated in the forcing region, and these propagate meridionally before dissipating thus broadening the dissipation distribution and allowing the generation of a mean flow.

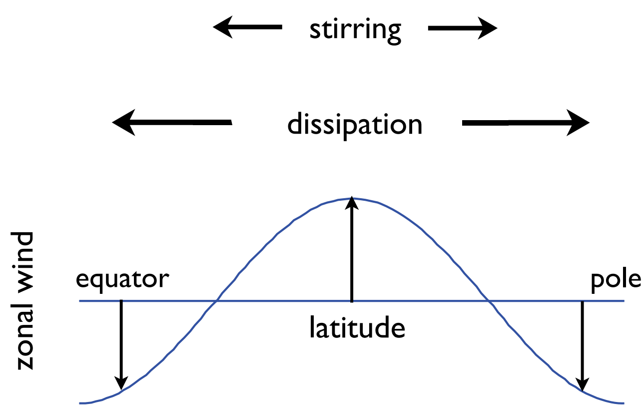


Figure 3.4 Mean flow generation by a meridionally confined stirring. Because of Rossby wave propagation away from the source region, the distribution of pseudomomentum dissipation is broader than that of pseudomomentum forcing, and the sum of the two leads to the zonal wind distribution shown, with positive (eastward) values in the region of the stirring. See also Fig. 3.6.

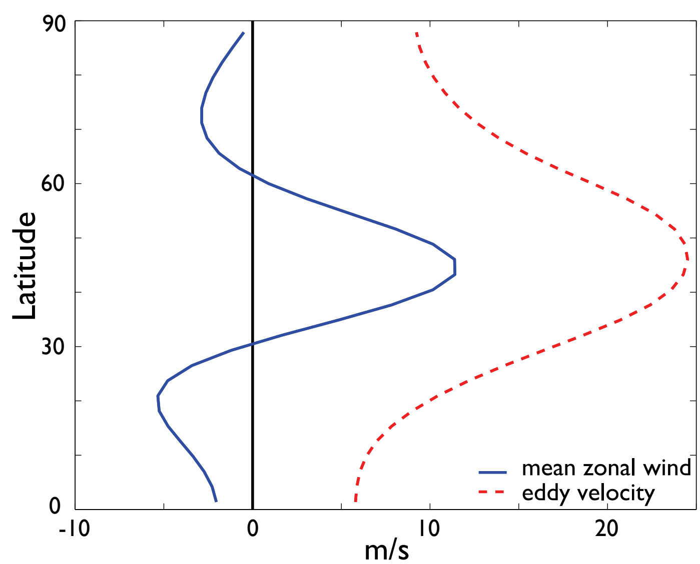


Figure 3.5 The time and zonally averaged wind (solid line) obtained by an integration of the barotropic vorticity equation on the sphere. The fluid is stirred in mid-latitudes by a random wavemaker that is statistically zonally uniform, acting around zonal wavenumber 8, and that supplies no net momentum. Momentum converges in the stirring region leading to an eastward jet with a westward flow to either side, and zero area-weighted spatially integrated velocity. The dashed line shows the r.m.s. (eddy) velocity created by the stirring.

3.5 Rossby Waves in an Inhomogeneous Medium

Consider the horizontal problem with infinite deformation radius and linearized equation of motion

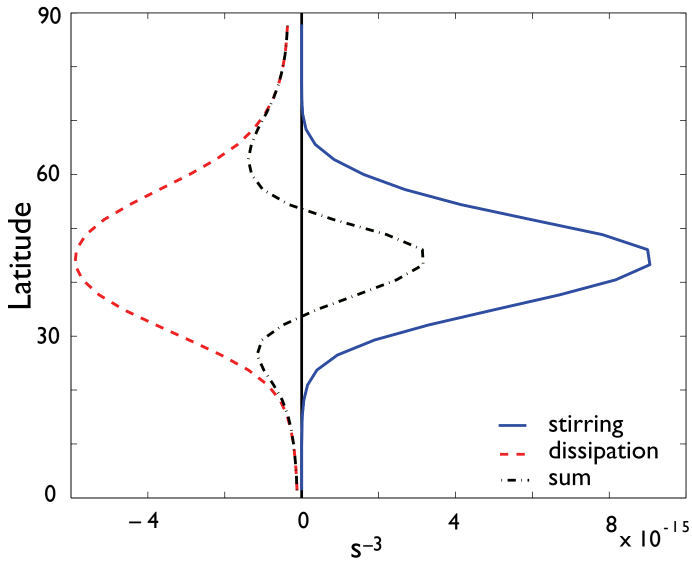
$$\left(\frac{\partial}{\partial t} + \overline{u}(y)\frac{\partial}{\partial x}\right)q' + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{3.37}$$

where $q' = \nabla^2\psi'$, $v' = \partial\psi'/\partial x$ and $\partial\overline{q}/\partial y = \beta - \overline{u}_{yy}$. If $\overline{u}$ and $\partial\overline{q}/\partial y$ do not vary in space then we may seek wavelike solutions in the usual way and obtain the dispersion relation

$$\omega \equiv ck = \overline{u}k - \frac{\partial\overline{q}/\partial y}{k}k^2 + l^2 \tag{3.38}$$

where k and l are the x- and y-wavenumbers.

Figure 3.6 The pseudomomentum stirring (solid line, $\overline{F'_\zeta \zeta'}$), dissipation (dashed line, $\overline{D'_\zeta \zeta'}$) and their sum (dot–dashed), for the same integration as Fig. 3.5. Because Rossby waves propagate away from the stirred region before breaking, the distribution of dissipation is broader than the forcing, resulting in an eastward jet where the stirring is centred, with westward flow on either side.



If the parameters do vary in the y-direction then we seek a solution of the form $\psi' = \widetilde{\psi}(y)\exp[ik(x - ct)]$ and obtain

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \quad \text{where} \quad l^2(y) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2 \tag{3.39a,b}$$

If the parameter variation is sufficiently small, occurring on a spatial scale longer than the wavelength of the waves, then we may expect that the disturbance will propagate locally as a plane wave. The solution is then of WKB form namely

$$\widetilde{\psi}(y) = A_0 l^{-1/2} \exp\left(i \int l \, dy\right). \tag{3.40}$$

where A_0 is a constant. The phase of the wave in the y-direction, θ, is evidently given by $\theta = \int l \, dy$, so that the local wavenumber is given by $d\theta/dy = l$. The group velocity is, as before,

$$c_g^x = \overline{u} + \frac{(k^2 - l^2)\partial \overline{q}/\partial y}{(k^2 + l^2)^2}, \qquad c_g^y = \frac{2kl \, \partial \overline{q}/\partial y}{(k^2 + l^2)^2}. \tag{3.41a,b}$$

The group velocity can now vary spatially, although it is only allowed to vary slowly.

3.5.1 Wave amplitude

As a Rossby wave propagates its amplitude is not necessarily constant because, in the presence of a shear, the wave may exchange energy with the background state. It goes like $l^{-1/2}(y)$. This variation can be understood from somewhat more general considerations. As we saw earlier in the simple one-layer case (and discussed more in the appendix) an inviscid, adiabatic wave will conserve its wave activity meaning that

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0, \tag{3.42}$$

where $\mathcal{A}$ is the wave amplitude and $\boldsymbol{\mathcal{F}}$ is the flux, and $\boldsymbol{\mathcal{F}} = \mathbf{c}_g \mathcal{A}$. In the stratified case we have

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \boldsymbol{\mathcal{F}} = -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k}, \tag{3.43}$$

with $\boldsymbol{\mathcal{F}}$ is the Eliassen–Palm (EP) flux, and in the 2D case there is no buoyancy and the $\mathbf{k}$ component is zero. If the waves are steady then $\nabla \cdot \boldsymbol{\mathcal{F}} = 0$, and in the two-dimensional case under consideration this means that $\partial \overline{u'v'}/\partial y = 0$.

Thus $u'v' = kl|\widetilde{\psi}|^2 =$ constant, and since k is constant the amplitude of a wave varies like

$$|\widetilde{\psi}| = \frac{A_0}{\sqrt{l(y)}} \tag{3.44}$$

as in the WKB solution. The energy of the wave then varies like

$$\text{Energy} = (k^2 + l^2)\frac{A_0^2}{l}. \tag{3.45}$$

3.5.2 Two examples

(i) Waves with a turning latitude

A turning line arises where $l = 0$. The line arises if the potential vorticity gradient diminishes to such an extent that $l^2 < 0$ and the waves then cease to propagate in the y-direction. This may happen even in in unsheared flow as a wave propagates polewards and the magnitude of beta diminishes.

As a wave packet approaches a turning latitude then l goes to zero so the amplitude, and the energy, of the wave approach infinity. This may happen as a wave propagates polewards and β diminishes. However, the wave will never reach the turning latitude because the meridional component of the group velocity is zero, as can be seen from the expressions for the group velocity, (3.41). As a wave approaches the turning latitude $c_g^x \to (\beta - \overline{u}_{yy})/k^2$ and $c_g^y \to 0$, so the group velocity is purely zonal and indeed as $l \to 0$

$$\frac{c_g^x - \overline{u}}{c_g^y} = \frac{k}{2l} \to \infty. \tag{3.46}$$

Because the meridional wavenumber is small the wavelength is large, so we do not expect the waves to break. Rather, we intuitively expect that a wave packet will turn — hence the eponym 'turning latitude' — and be reflected.

Rossby Wave Propagation in a Slowly Varying Medium

The linear equation of motion is, in terms of streamfunction,

$$\left(\frac{\partial}{\partial t} + \overline{u}(y,z)\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\frac{\partial \overline{q}}{\partial y} = 0. \qquad \text{(RP.1)}$$

We suppose that the parameters of the problem vary slowly in y and/or z but are uniform in x and t. The frequency and zonal wavenumber are therefore constant. We seek solutions of the form $\psi' = \widetilde{\psi}(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find (if, for simplicity, N^2 and ρ_R are constant)

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2\widetilde{\psi}}{\partial z^2} + n^2(y,z)\widetilde{\psi} = 0 \qquad \text{(RP.2a)}$$

where

$$n^2(y,z) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2. \qquad \text{(RP.2b)}$$

The value of n^2 must be positive in order that waves can propagate, and so waves cease to propagate when they encounter either

1. *A turning line*, where $n^2 = 0$, or
2. *A critical line*, where $\overline{u} = c$ and n^2 becomes infinite.

The bounds may usefully be expressed as a condition on the zonal flow:

$$0 < \overline{u} - c < \frac{\partial \overline{q}/\partial y}{k^2}. \qquad \text{(RP.3)}$$

If the length scale over which the parameters of the problem vary is much longer than the wavelengths themselves we can expect the solution to look locally like a plane wave and a WKB analysis can be employed. In the purely horizontal problem we assume a solution of the form $\psi' = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \qquad l^2(y) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2. \qquad \text{(RP.4)}$$

The solution is of the form

$$\widetilde{\psi}(y) = A l^{-1/2} \exp\left(\pm \mathrm{i}\int l \,\mathrm{d}y\right). \qquad \text{(RP.5)}$$

Thus, $l(y)$ is the local y-wavenumber, and the amplitude of the solution varies like $l^{-1/2}$. At a critical line the amplitude of the wave will go to zero although the energy may become very large, and since the wavelength is small the waves may break. At a turning line the amplitude and energy will both be large, but since the wavelength is long the waves will not necessarily break. A similar analysis may be employed for vertically propagating Rossby waves.

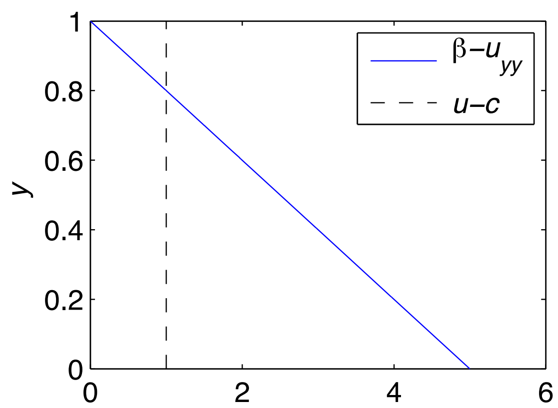


Figure 3.7 Parameters for the first example considered in section 3.5.2, with all variables nondimensional. The zonal flow is uniform with $u = 1$ and $c = 0$ (so that $\overline{u}_{yy} = 0$) and β diminishes linearly as y increases polewards as shown. With zonal wavenumber $k = 1$ there is a turning latitude at $y = 0.8$, and the wave properties are illustrated in Fig. 3.8.

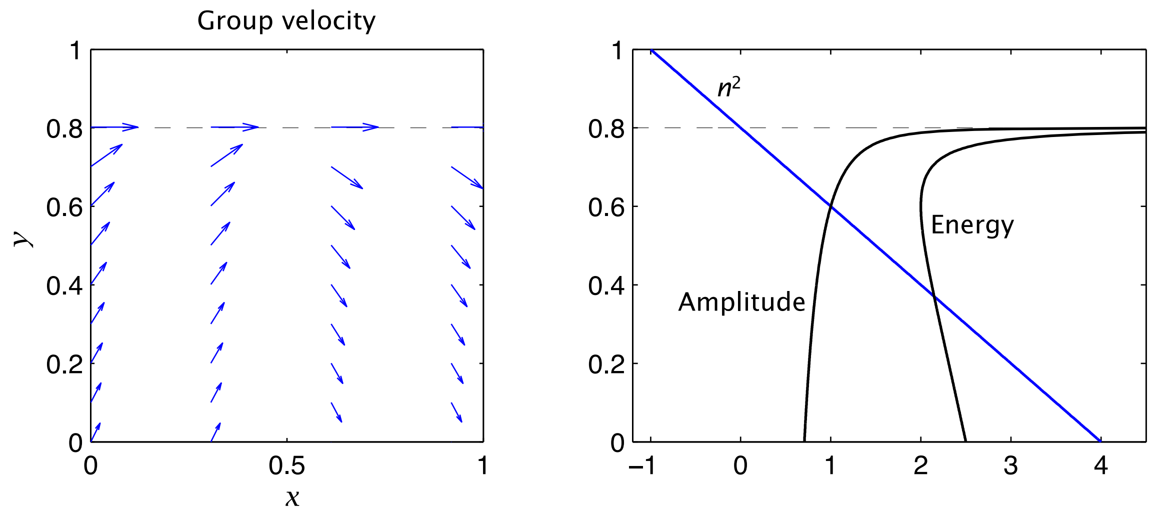
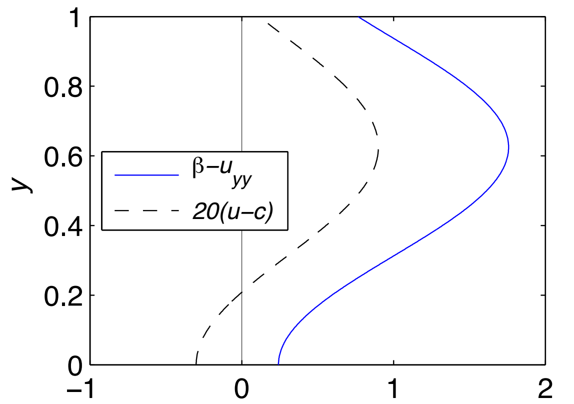


Figure 3.8 Left: The group velocity evaluated using (3.41) for the parameters illustrated in Fig. 3.7, which give a turning latitude at $y = 0.8$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared (n^2), the energy and the amplitude of a wave. n^2 is negative for $y > 0.8$. See text for more description.

To illustrate this, consider waves propagating in a background state that has a beta effect that diminishes polewards but no horizontal shear. To be concrete suppose that $\beta = 5$ at $y = 0$, diminishing linearly to $\beta = 0$ at $y = 0$, and that $\overline{u} - c = 1$ everywhere. There is no critical line but depending on the x-wavenumber there may be a turning line, and if we choose $k = 1$ then the turning line occurs when $\beta = 1$ and so at $y = 0.8$. Note that the turning latitude depends on the value of the x-wavenumber — if the zonal wavenumber is larger then waves will turn further south. The parameters are illustrated in Fig. 3.7.

For a given zonal wavenumber ($k = 1$ in this example) the value of l^2 is computed using

Figure 3.9 Parameters for the second example considered in section 3.5.2, with all variables nondimensional. The zonal flow has a broad eastward jet and β is constant. There is a critical line at $y = 0.2$, and with zonal wavenumber $k = 5$ the wave properties are illustrated in Fig. 3.10.



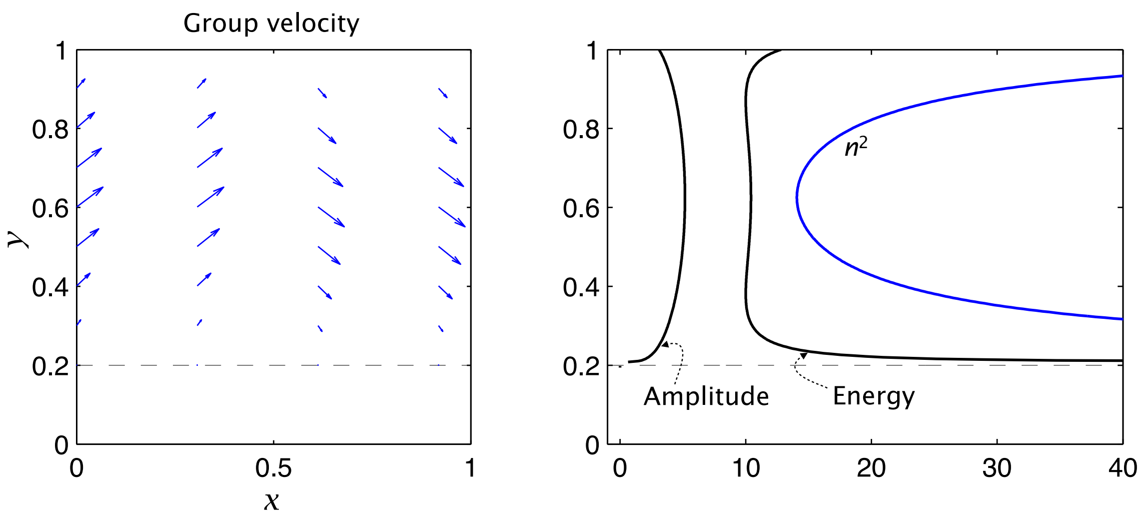


Figure 3.10 Left: The group velocity evaluated using (3.41) for the parameters illustrated in Fig. 3.7, which give a critical line at $y = 0.2$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared, the energy and the amplitude of a wave. The value of n^2 becomes infinite at the critical line. See text for more description.

(3.39b), and the components of the group velocity using (3.41), and these are illustrated in Fig. 3.8. Note that we may choose either a positive or a negative value of l, corresponding to northward or southward oriented waves, and we illustrate both in the figure. The value of l^2 becomes zero at $y = 0.8$, and this corresponds to a turning latitude. The values of the wave amplitude and energy are computed using (3.44) and (3.45) (with an arbitrary amplitude at $y = 0$) and these both become infinite at the turning latitude.

(ii) Waves with a critical latitude

A critical line occurs when $\overline{u} = c$, corresponding to the upper bound of c, and from (3.39) we see that at a critical line the meridional wavenumber approaches infinity. From (3.41) we see that both the x- and y-components of the group velocity are zero — a wave packet approaching a critical line just stops. Specifically, as l becomes large

$$c_g^x - \overline{u} \to 0, \qquad c_g^y \to 0, \qquad \frac{c_g^x - \overline{u}}{c_g^y} \to -\frac{l}{k} \to -\infty. \tag{3.47}$$

From (3.44) the amplitude of the wave packet also approaches zero, but its energy approaches infinity. Since the wavelength is very small we expect the waves to *break* and deposit their momentum, and this situation commonly arises when Rossby waves excited in midlatitudes propagate equatorward and encounter a critical latitude in the subtropics.

To illustrate this let us construct background state that has an eastward jet in midlatitudes becoming westward at low latitudes, with β constant chosen to be large enough so that $\beta - \overline{u}_{yy}$ is positive everywhere. (Specifically, we choose $\beta = 1$ and $\overline{u} = -0.03\sin(8\pi y/5+\pi/2)-0.5$), but the precise form is not important.) If $c = 0$ then there is a critical line when $\overline{u}$ passes through zero, which in this example occurs at $x = 0.2$. (The value of $\overline{u} - c$ is small at $y = 1$, but no critical line is actually reached.) These parameters are illustrated in Fig. 3.9. We also choose $k = 5$, which results in a positive value for l^2 everywhere.

As in the previous example we compute the value of l^2 using (3.39b) and the components of the group velocity using (3.41), and these are illustrated in Fig. 3.10, with northward propagating waves shown for $x < 0.5$ and southward propagating waves for $x > 0.5$. The value of l^2 increases considerably at the northern and southern edges of the domain, and is actually infinite at the critical line at $y = 0.2$. Using (3.44) the amplitude of the wave diminishes as the critical line approaches, but the energy increases rapidly, suggesting that the linear approximation will break down. The waves will actually stall before reaching the critical layer, because both the x and the y components of the group velocity become very small. Also, because the wavelength is so small we may expect the waves to break and deposit their momentum, but a full treatment of waves in the vicinity of a critical layer requires a nonlinear analysis.

The situation illustrated in this example is of particular relevance to the maintenance of the zonal wind structure in the troposphere. Waves are generated in midlatitude and propagate equatorward and on encountering a critical layer in the subtropics they break, deposit westward momentum and retard the flow, as the reader who braves the next section will discover explicitly.

3.6 Rossby Wave Absorption near a Critical Layer

We noted in the last section that as a wave approaches a critical latitude the meridional wavenumber l becomes very large, but the group velocity itself becomes small. These observations suggest that the effects of friction might become very large and that the wave would deposit its momentum, thereby accelerating or decelerating the mean flow, and if we are willing to make one or two approximations we can construct an explicit analytic model of this phenomena. Specifically, we will need to choose a simple form for the friction and assume that the background properties vary slowly, so that we can use a WKB approximation. Note that we have to include some form of dissipation, otherwise the Eliassen–Palm flux divergence is zero and there is no momentum deposition by the waves.

3.6.1 A model problem

Consider horizontally propagating Rossby waves obeying the linear barotropic vorticity equation on the beta-plane (vertically propagating waves may be considered using similar techniques). The equation of motion is

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta^*\frac{\partial \psi}{\partial x} = -r\nabla^2\psi, \tag{3.48}$$

where $\beta^* = \beta - \overline{u}_{yy}$. The parameter r is a drag coefficient that acts directly on the relative vorticity. It is not a particularly realistic form of dissipation but its simplicity will serve our purpose well. We shall assume that r is small compared to the Doppler-shifted frequency of the waves and seek solutions of the form

$$\psi'(x, y, t) = \widetilde{\psi}(y)e^{i(k(x-ct))}. \tag{3.49}$$

Substituting into (3.48) we find, after a couple of lines of algebra, that $\widetilde{\psi}$ satisfies, analogously to (3.39),

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \quad \text{where} \quad l^2(y) = \frac{\beta^*}{\overline{u} - c - ir/k} - k^2. \tag{3.50a,b}$$

Evidently, as with the inviscid case, if the zonal wind has a lateral shear then l is a function of y. However, l now has an imaginary component so that the wave decays away from its source region. We can already see that if $\overline{u} = c$ the decay will be particularly strong.

3.6.2 WKB solution

Let us suppose that the zonal wavenumber is small compared to the meridional wavenumber l, which will certainly be the case approaching a critical layer. If $r \ll k(\overline{u}-c)$ then the meridional wavenumber is given by

$$l^2(y) \approx \left[\frac{\beta^*(\overline{u} - c + ir/k)}{(\overline{u} - c)^2 + r^2/k^2}\right] \approx \frac{\beta^*}{\overline{u} - c}\left[1 + \frac{ir}{k(\overline{u} - c)}\right] \tag{3.51}$$

whence

$$l(y) \approx \left(\frac{\beta^*}{\overline{u} - c}\right)^{1/2}\left[1 + \frac{ir}{2k(\overline{u} - c)}\right]. \tag{3.52}$$

The streamfunction itself is then given by, in the WKB approximation,

$$\widetilde{\psi} = A l^{-1/2} \exp\left(\pm \mathrm{i} \int^{y} l \,\mathrm{d}y' \right). \tag{3.53}$$

But now the wave will decay as it moves away from its source and deposit momentum into the mean flow, as we now calculate.

The momentum flux, F_k, associated with the wave with x-wavenumber of k is given by

$$F_k(y) = \overline{u'v'} = -\mathrm{i}k \left(\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y} \right), \tag{3.54}$$

and using (3.52) and (3.53) in (3.54) we obtain

$$F_k(y) = F_0 \exp\left(\pm \mathrm{i} \int_0^y (l - l^*) \,\mathrm{d}y' \right) = F_0 \exp\left(\left(\int_0^y \frac{\pm r \beta^{*1/2}}{k(\overline{u} - c)^{3/2}} \,\mathrm{d}y' \right) \right). \tag{3.55}$$

In deriving this expression we use that fact that the amplitude of $\widetilde{\psi}$ (i.e., $l^{-1/2}$) varies only slowly with y so that when calculating $\partial \widetilde{\psi} / \partial y$ its derivative may be ignored. In (3.55) F_0 is the value of the flux at $y = 0$ and the sign of the exponent must be chosen so that the group velocity is directed away from the wave source region. Clearly, if $r = 0$ then the momentum flux is constant.

The integrand in (3.55) is the attenuation rate of the wave and it has a straightforward physical interpretation. Using the real part of (3.52) in (3.41b), and assuming $|l| \gg |k|$, the meridional component of the group velocity is given by

$$c_g^y = \frac{2kl\,\beta^*}{(k^2 + l^2)^2} \approx \frac{2k\,\beta^*}{l^3} = \frac{2k(\overline{u} - c)^{3/2}}{\beta^{*1/2}}. \tag{3.56a,b}$$

Thus we have

$$\text{Wave attenuation rate} = \frac{r\beta^{*1/2}}{k(\overline{u} - c)^{3/2}} = \frac{2 \times \text{Dissipation rate, } 2r}{\text{Meridional group velocity, } c_g^y}. \tag{3.57}$$

As the group velocity diminishes the dissipation has more time to act and so the wave is preferentially attenuated, a result that we discuss more in the next subsection.

How does this attenuation affect the mean flow? The mean flow is subject to many waves and so obeys the equation

$$\frac{\partial \overline{u}}{\partial t} = -\sum_k \frac{\partial F_k}{\partial y} + \text{viscous terms}. \tag{3.58}$$

Because the amplitude varies only slowly compared to the phase, the amplitude of $\partial F_k / \partial y$ varies mainly with the attenuation rate (3.57) and is largest near a critical layer. Consider a Rossby wave propagating away from some source region with a given frequency and x-wavenumber.

Because k is negative a Rossby wave always carries westward (or negative) momentum with it. That is, F_k is always negative and increases (becomes more positive) as the wave is attenuated; that is to say, if $r \neq 0$ then $\partial F_k/\partial y$ is positive and from (3.58) the mean flow is accelerated *westward* as the wave dissipates. This acceleration will be particularly strong if the wave approaches a critical layer where $\overline{u} = c$. Indeed, such a situation arises when Rossby waves, generated in mid-latitudes, propagate equatorward. As the waves enter the subtropics $\overline{u} - c$ becomes smaller and the waves dissipate, producing a westward force on the mean flow, even though a true critical layer may never be reached. Globally, momentum is conserved because there is an equal and opposite (and therefore eastward) wave force at the wave source producing an eddy-driven jet, as discussed in the previous chapter.

3.6.3 Interpretation using wave activity

We can derive and interpret the above results by thinking about the propagation of wave activity. For barotropic Rossby waves, multiply (3.48) by ζ/β^* and zonally average to obtain the wave activity equation,

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial \mathcal{F}}{\partial y} = -\alpha \mathcal{A}, \tag{3.59}$$

where $\mathcal{A} = \overline{\zeta'^2}/2\beta^*$ is the wave activity density, $\partial \mathcal{F}/\partial y = \overline{v'\zeta'}$ is its flux divergence, and $\alpha = 2r$. Referring as needed to the discussion in sections 3.A.2 and 3.A.3, the flux obeys the group velocity property so that

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial y}(\boldsymbol{c}_g \mathcal{A}) = -\alpha \mathcal{A}. \tag{3.60}$$

Let us suppose that the wave is in a statistical steady state and that the spatial variation of the group velocity occurs on a longer spatial scale than the variations in wave activity density, consistent with the WKB approximation. We then have

$$c_g^y \frac{\partial \mathcal{A}}{\partial y} = -\alpha \mathcal{A}. \tag{3.61}$$

which integrates to give

$$\mathcal{A}(y) = \mathcal{A}_0 \exp\left(- \int^y \frac{\alpha}{c_g^y} \, \mathrm{d}y' \right). \tag{3.62}$$

That is, the attenuation rate of the wave activity is the dissipation rate of wave activity divided by the group velocity, as in (3.55) and (3.57) (note that $\alpha = 2r$).

3.A APPENDIX A: VARIOUS PROPERTIES OF ROSSBY WAVES

In this appendix we derive various properties of Rossby waves useful in wave–mean-flow interaction theory, assuming a good knowledge of stratified quasi-geostrophic theory. We use the Boussinesq approximation throughout. This material was not presented in the lectures at Walsh.

3.A.1 The Eliassen–Palm Flux

The eddy flux of potential vorticity may be expressed in terms of vorticity and buoyancy fluxes as

$$v'q' = v'\zeta' + f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right). \tag{3.63}$$

The second term on the right-hand side can be written as

$$\begin{aligned} f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right) &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial v'}{\partial z}\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - \frac{f_0^2}{2N^2}\frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)^2, \end{aligned} \tag{3.64}$$

using $b' = f_0 \partial\psi'/\partial z$.

Similarly, the flux of relative vorticity can be written

$$v'\zeta' = -\frac{\partial}{\partial y}(u'v') + \frac{1}{2}\frac{\partial}{\partial x}(v'^2 - u'^2) \tag{3.65}$$

Using (3.64) and (3.65), (3.63) becomes

$$\boxed{v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{1}{2}\frac{\partial}{\partial x}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right).} \tag{3.66}$$

Thus the meridional potential vorticity flux, in the quasi-geostrophic approximation, can be written as the divergence of a vector: $v'q' = \nabla \cdot \boldsymbol{\mathcal{E}}$ where

$$\boldsymbol{\mathcal{E}} \equiv \frac{1}{2}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right)\mathbf{i} - (u'v')\,\mathbf{j} + \left(\frac{f_0}{N^2}v'b'\right)\mathbf{k}. \tag{3.67}$$

A particularly useful form of this arises after zonally averaging, for then (3.66) becomes

$$\boxed{\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right)}. \tag{3.68}$$

The vector defined by

$$\boldsymbol{\mathcal{F}} \equiv -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k} \tag{3.69}$$

is called the (quasi-geostrophic) *Eliassen–Palm (EP) flux*,[1] and its divergence, given by (3.68), gives the poleward flux of potential vorticity:

$$\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}, \tag{3.70}$$

where $\nabla_x \cdot \equiv (\partial/\partial y, \partial/\partial z)\cdot$ is the divergence in the meridional plane. Unless the meaning is unclear, the subscript x on the meridional divergence will be dropped.

3.A.2 The Eliassen–Palm relation

On dividing by $\partial\overline{q}/\partial y$ and using (3.70), the enstrophy equation (**??**) becomes

$$\boxed{\frac{\partial \mathcal{A}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \mathcal{D}}, \tag{3.71a}$$

where

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \mathcal{D} = \frac{\overline{D'q'}}{\partial\overline{q}/\partial y}. \tag{3.71b}$$

Equation (3.71a) is known as the *Eliassen–Palm relation,* and it is a conservation law for the *wave activity density* $\mathcal{A}$. The conservation law is exact (in the linear approximation) if the mean flow is constant in time. It will be a good approximation if $\partial\overline{q}/\partial y$ varies slowly compared to the variation of $\overline{q'^2}$.

If we integrate (3.71b) over a meridional area A bounded by walls where the eddy activity vanishes, and if $\mathcal{D} = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{A}\,\mathrm{d}A = 0. \tag{3.72}$$

The integral is a wave activity — a quantity that is quadratic in the amplitude of the perturbation and that is conserved in the absence of forcing and dissipation. In this case $\mathcal{A}$ is the negative of the *pseudomomentum,* for reasons we will encounter later. ('Wave action' is a particular form of wave activity; it is the energy divided by the frequency and it is a conserved property in many wave problems.) Note that neither the perturbation energy nor the perturbation enstrophy are wave activities of the linearized equations, because there can be an exchange of energy or enstrophy between mean and perturbation — indeed, this is how a perturbation grows in baroclinic or barotropic instability! This is already evident from (**??**), or in general take (**??**) with $D' = 0$ and multiply by q' to give the enstrophy equation

$$\frac{1}{2}\frac{\partial q'^2}{\partial t} + \frac{1}{2}\overline{\boldsymbol{u}}\cdot\nabla q'^2 + \boldsymbol{u}'q'\cdot\nabla\overline{q} = 0, \tag{3.73}$$

where here the overbar is an average (although it need not be a zonal average). Integrating this over a volume V gives

$$\frac{\mathrm{d}\hat{Z}'}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}q'^2\,\mathrm{d}V = -\int_V \boldsymbol{u}'q'\cdot\nabla\overline{q}\,\mathrm{d}V. \tag{3.74}$$

The right-hand side does not, in general, vanish and so $\hat{Z}'$ is not in general conserved.

3.A.3 The group velocity property for Rossby waves

The vector $\boldsymbol{\mathcal{F}}$ describes how the wave activity propagates. In the case in which the disturbance is composed of plane or almost plane waves that satisfy a dispersion relation, then $\boldsymbol{\mathcal{F}} = \boldsymbol{c}_g\mathcal{A}$, where $\boldsymbol{c}_g$ is the group velocity and (3.71a) becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot (\mathcal{A}\boldsymbol{c}_g) = 0. \tag{3.75}$$

This is a useful property, because if we can diagnose $\boldsymbol{c}_g$ from observations we can use (3.71a) to determine how wave activity density propagates. Let us demonstrate this explicitly for the pseudomomentum in Rossby waves, that is for (3.71a).

The Boussinesq quasi-geostrophic equation on the β-plane, linearized around a uniform zonal flow and with constant static stability, is

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{3.76}$$

where $q' = [\nabla^2 + (f_0^2/N^2)\partial^2/\partial z^2]\psi'$ and, if $\overline{u}$ is constant, $\partial\overline{q}/\partial y = \beta$. Thus we have

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \beta\frac{\partial \psi'}{\partial x} = 0. \tag{3.77}$$

Seeking solutions of the form

$$\psi' = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)}, \tag{3.78}$$

we find the dispersion relation,

$$\omega = \overline{u}k - \frac{\beta k}{\kappa^2}. \tag{3.79}$$

where $\kappa^2 = (k^2 + l^2 + m^2 f_0^2/N^2)$, and the group velocity components:

$$c_g^y = \frac{2\beta kl}{\kappa^4}, \qquad c_g^z = \frac{2\beta km f_0^2/N^2}{\kappa^4}. \tag{3.80}$$

Also, if $u' = \mathrm{Re}\,\widetilde{u}\exp[\mathrm{i}(kx + ly + mz - \omega t)]$, and similarly for the other fields, then

$$\begin{aligned} \widetilde{u} &= -\mathrm{Re}\ \mathrm{i}l\widetilde{\psi}, \qquad & \widetilde{v} &= \mathrm{Re}\ \mathrm{i}k\widetilde{\psi}, \\ \widetilde{b} &= \mathrm{Re}\,\mathrm{i}mf_0\widetilde{\psi}, \qquad & \widetilde{q} &= -\mathrm{Re}\,\kappa^2\widetilde{\psi}, \end{aligned} \tag{3.81}$$

The wave activity density is then

$$\mathcal{A} = \frac{1}{2}\frac{\overline{q'^2}}{\beta} = \frac{\kappa^4}{4\beta}|\widetilde{\psi}^2|, \tag{3.82}$$

where the additional factor of 2 in the denominator arises from the averaging. Using (3.81) the EP flux, (3.69), is

$$\mathcal{F}^y = -\overline{u'v'} = \frac{1}{2}kl|\widetilde{\psi}^2|, \qquad \mathcal{F}^z = \frac{f_0}{N^2}\overline{v'b'} = \frac{f_0^2}{2N^2}km|\widetilde{\psi}^2|. \tag{3.83}$$

Using (3.80), (3.82) and (3.83) we obtain

$$\boxed{\boldsymbol{\mathcal{F}} = (\mathcal{F}^y, \mathcal{F}^z) = \boldsymbol{c}_g \mathcal{A}}. \tag{3.84}$$

If the properties of the medium are slowly varying, so that a (spatially varying) group velocity can still be defined, then this is a useful expression to estimate how the wave activity propagates in the atmosphere and in numerical simulations.

3.A.4 Energy flux in Rossby waves

Start with

$$\frac{\partial}{\partial t}\left(\nabla^2 - k_d^2\right)\psi + \beta \frac{\partial \psi}{\partial x} = 0. \tag{3.85}$$

To obtain an energy equation multiply (3.85) by $-\psi$ and obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left((\nabla\psi)^2 + k_d^2\psi^2\right) - \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = 0, \tag{3.86}$$

where $\mathbf{i}$ is the unit vector in the x direction. The first group of terms are the energy itself, or more strictly the energy density. (An energy density is an energy per unit mass or per unit volume, depending on the context.) The term $(\nabla\psi)^2/2 = (u^2 + v^2)/2$ is the kinetic energy and $k_d^2\psi^2/2$ is the potential energy, proportional to the displacement of the free surface, squared. The second term is the energy flux, so that we may write

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = 0. \tag{3.87}$$

where $E = (\nabla\psi)^2/2 + k_d^2\psi^2$ and $\boldsymbol{F} = -\left(\psi\nabla\partial\psi/\partial t + \mathbf{i}\beta\psi^2\right)$. We haven't yet used the fact that the disturbance has a dispersion relation, and if we do so we may expect that the energy moves at the group velocity. Let us now demonstrate this explicitly.

We assume a solution of the form

$$\psi = A(x)\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) = A(x)\cos(kx + ly - \omega t) \tag{3.88}$$

where $A(x)$ is assumed to vary slowly compared to the nearly plane wave. (Note that $\boldsymbol{k}$ is the wave vector, to be distinguished from $\mathbf{k}$, the unit vector in the z-direction.) The kinetic energy in a wave is given by

$$KE = \frac{A^2}{2}\left(\psi_x^2 + \psi_y^2\right) \tag{3.89}$$

so that, averaged over a wave period,

$$\overline{KE} = \frac{A^2}{2}(k^2 + l^2)\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\,\mathrm{d}t. \tag{3.90}$$

The time-averaging produces a factor of one half, and applying a similar procedure to the potential energy we obtain

$$\overline{KE} = \frac{A^2}{4}(k^2 + l^2), \qquad \overline{PE} = \frac{A^2}{4}k_d^2, \tag{3.91}$$

so that the average total energy is

$$\overline{E} = \frac{A^2}{4}(K^2 + k_d^2), \tag{3.92}$$

where $K^2 = k^2 + l^2$.

The flux, $\boldsymbol{F}$, is given by

$$\boldsymbol{F} = -\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = -A^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\left(\boldsymbol{k}\omega - \mathbf{i}\frac{\beta}{2}\right), \tag{3.93}$$

so that evidently the energy flux has a component in the direction of the wavevector, $\boldsymbol{k}$, and a component in the x-direction. Averaging over a wave period straightforwardly gives us additional factors of one half:

$$\overline{\boldsymbol{F}} = -\frac{A^2}{2}\left(\boldsymbol{k}\omega + \mathbf{i}\frac{\beta}{2}\right). \tag{3.94}$$

We now use the dispersion relation $\omega = -\beta k/(K^2 + k_d^2)$ to eliminate the frequency, giving

$$\overline{\boldsymbol{F}} = \frac{A^2\beta}{2}\left(\boldsymbol{k}\frac{k}{K^2 + k_d^2} - \mathbf{i}\frac{1}{2}\right), \tag{3.95}$$

and writing this in component form we obtain

$$\overline{\boldsymbol{F}} = \frac{A^2\beta}{4}\left[\mathbf{i}\left(\frac{k^2 - l^2 - k_d^2}{K^2 + k_d^2}\right) + \mathbf{j}\left(\frac{2kl}{K^2 + k_d^2}\right)\right] \tag{3.96}$$

Comparison of (3.96) with (3.15) and (3.92) reveals that

$$\overline{\boldsymbol{F}} = \boldsymbol{c}_g\overline{E} \tag{3.97}$$

so that the energy propagation equation, (3.87), when averaged over a wave, becomes

$$\frac{\partial\overline{E}}{\partial t} + \nabla\cdot\boldsymbol{c}_g\overline{E} = 0. \tag{3.98}$$

This is an important result, and more general than our derivation implies. One immediate implication is that if there is a disturbance that generates waves, *the group velocity is directed away from the disturbance.*

Most of the time in waves, energy is not conserved because it can be extracted from the flow.

3.B Appendix B: The WKB Approximation for Linear Waves

We are concerned with finding solutions to an equation of the form

$$\frac{d^2\xi}{dz^2} + m^2(z)\xi = 0, \tag{3.99}$$

where $m^2(z)$ is positive for wavelike solutions. If m is constant the solution has the harmonic form

$$\xi = \mathrm{Re}\, A_0 e^{imz} \tag{3.100}$$

where A_0 is a complex constant. If m varies only 'slowly' with z — meaning that the variations occur on a scale much longer than $1/m$ — one might reasonably expect that the harmonic solution above would provide a reasonable first approximation; that is, we expect the solution to locally look like a plane wave with local wavenumber $m(z)$. However, we might also expect that the solution would not be *exactly* of the form $\exp(im(z)z)$, because the phase of ξ is $\theta(z) = mz$, so that $d\theta/dz = m + z\,dm/dz \neq m$. Thus, in (3.100) m is not the wavenumber unless m is constant. Nevertheless, this argument suggests that we seek solutions of a similar form to (3.100), and we find such solutions by way of a perturbation expansion below. We note that the condition that variations in m, or in the wavelength m^{-1}, occur only slowly may be expressed as

$$\frac{m}{|\partial m/\partial z|} \gg m^{-1} \quad \text{or} \quad \left|\frac{\partial m}{\partial z}\right| \ll m^2. \tag{3.101}$$

This condition will generally be satisfied if variations in the background state, or in the medium, occur on a scale much longer than the wavelength.

3.B.1 Solution by perturbation expansion

To explicitly recognize the rapid variation of m we rescale the coordinate z with a small parameter ϵ; that is, we let $\hat{z} = \epsilon z$ where $\hat{z}$ varies by $\mathcal{O}(1)$ over the scale on which m varies. Eq. (3.99) becomes

$$\epsilon^2 \frac{d^2\xi}{d\hat{z}^2} + m^2(\hat{z})\xi = 0, \tag{3.102}$$

and we may now suppose that all variables are $\mathcal{O}(1)$. If m were constant the solution would be of the form $\xi = A\exp(m\hat{z}/\epsilon)$ and this suggests that we look for a solution to (3.102) of the form

$$\xi(z) = e^{g(\hat{z})/\epsilon}, \tag{3.103}$$

where $g(\hat{z})$ is some as yet unknown function. We then have, with primes denoting derivatives,

$$\xi' = \frac{1}{\epsilon} g' e^{g/\epsilon}, \qquad \xi'' = \left(\frac{1}{\epsilon^2} g'^2 + \frac{1}{\epsilon} g''\right) e^{g/\epsilon}. \tag{3.104a,b}$$

Using these expressions in (3.102) yields

$$\epsilon g'' + g'^2 + m^2 = 0, \tag{3.105}$$

and if we let $g = \int h \, d\hat{z}$ we obtain

$$\epsilon \frac{dh}{d\hat{z}} + h^2 + m^2 = 0. \tag{3.106}$$

To obtain a solution of this equation we expand h in powers of the small parameter ϵ,

$$h(\hat{z};\epsilon) = h_0(\hat{z}) + \epsilon h_1(\hat{z}) + \epsilon^2 h_2(\hat{z}) + \cdots . \tag{3.107}$$

Substituting this in (3.106) and setting successive powers of ϵ to zero gives, at first and second order,

$$h_0^2 + m^2 = 0, \qquad 2h_0 h_1 + \frac{dh_0}{d\hat{z}} = 0. \tag{3.108a,b}$$

The solutions of these equations are

$$h_0 = \pm i m, \qquad h_1(\hat{z}) = -\frac{1}{2}\frac{d}{d\hat{z}} \ln \frac{m(\hat{z})}{m_0}. \tag{3.109a,b}$$

where m_0 is a constant. Now, ignoring higher-order terms, (3.103) may be written in terms of h_0 and h_1 as

$$\xi(\hat{z}) = \exp\left(\int h_0 \, d\hat{z}/\epsilon\right) \exp\left(\int h_1 \, d\hat{z}\right), \tag{3.110}$$

and, using (3.109) and with z in place of $\hat{z}$, we obtain

$$\boxed{\xi(z) = A_0 m^{-1/2} \exp\left(\pm i \int m \, dz\right)}. \tag{3.111}$$

where A_0 is a constant, and this is the WKB solution to (3.99). In general

$$\xi(z) = B_0 m^{-1/2} \exp\left(i \int m \, dz\right) + C_0 m^{-1/2} \exp\left(-i \int m \, dz\right). \tag{3.112}$$

or

$$\xi(z) = D_0 m^{-1/2} \cos\left(\int m \, dz\right) + E_0 m^{-1/2} \sin\left(\int m \, dz\right). \tag{3.113}$$

A property of (3.111) is that the derivative of the phase is just m; that is, m is indeed the local wavenumber. Note that a crucial aspect of the derivation is that m varies slowly, so that there is a small parameter, ϵ, in the problem. Having said this, it is often the case that WKB theory can provide qualitative guidance even when there is little scale separation between the variation of the background state and the wavelength. Asymptotics often works when it seemingly shouldn't.

Chapter 4

The Hadley Cell

In this short chapter we take a look at the general circulation of the atmosphere, and in particular the Hadley Cell. Look again at the zonally averaged circulation in the top panel of Fig. 1.2. The centre two circulations are the Hadley cells. Deep tropical convection lifts air near at the Intertropical Convergence Zone (ITCZ) near the equator. At the tropopause its vertical motion is inhibited by strong static stability, so it begins a poleward migration which extends as far as some critical latitude ϑ_H. In this chapter we will attempt to explain *why* the Hadley cell terminates at ϑ_H, and not some other latitude. We will outline three possibilities.

1. The Hadley Cell is terminated in order to satisfy certain thermodynamic constraints, described in 4.1
2. The Hadley Cell is terminated by the onset of baroclinic instability, described in section 4.2.
3. The Hadley Cell is terminated by the effects of the breaking of Rossby waves, described in section 4.3.

Almost certainly none of these models describes the real Hadley Cell in anything other than an approximate way, but this does not mean they are not useful.

4.1 A Zonally Symmetric Steady Model of the Hadley Cell

We begin with a a model of the zonally symmetric circulation – that is, the circulation has no eddies, in fact no variation at all in the zonal direction. A parcel of air moving polewards away from the boundary layer will then conserve its axial angular momentum, as shown in Figure 4.1. To construct a mathematical model, following Schneider & Lindzen (1977) and Held & Hou (1980), we suppose the following.

(i) The circulation is steady.

(ii) The polewards moving air conserves its axial angular momentum, whereas the zonal flow associated with the near-surface, equatorwards moving flow is frictionally retarded and weak.

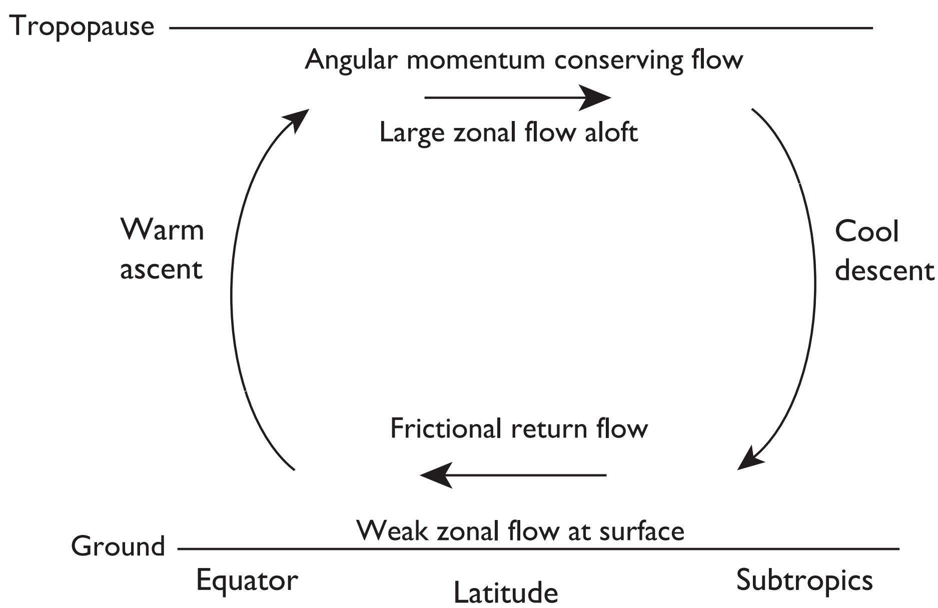


Figure 4.1 A simple model of the Hadley Cell. Rising air near the equator moves polewards near the tropopause, descending in the subtropics and returning near the surface. The polewards moving air conserves its axial angular momentum, leading to a zonal flow that increases away from the equator. By the thermal wind relation the temperature of the air falls as it moves polewards, and to satisfy the thermodynamic budget it sinks in the subtropics. The return flow at the surface is frictionally retarded and small.

(iii) The circulation is in thermal wind balance.

(iv) The flow is symmetric about the equator. Seasons can in fact be added to such a model.

4.1.1 Angular momentum conservation

Momentum equation:

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{a\cos^2\vartheta}\frac{\partial}{\partial \vartheta}(\cos^2\vartheta\,\overline{u'v'}) - \frac{\partial \overline{u'w'}}{\partial z}, \tag{4.1}$$

where $\overline{\zeta} = -(a\cos\vartheta)^{-1}\partial_\vartheta(\overline{u}\cos\vartheta)$ and the overbars represent zonal averages. We simplify this to

$$(f + \overline{\zeta})\overline{v} = 0. \tag{4.2}$$

It is easy to show that this is equivalent to

$$2\Omega\sin\vartheta = \frac{1}{a}\frac{\partial \overline{u}}{\partial \vartheta} - \frac{\overline{u}\tan\vartheta}{a}. \tag{4.3}$$

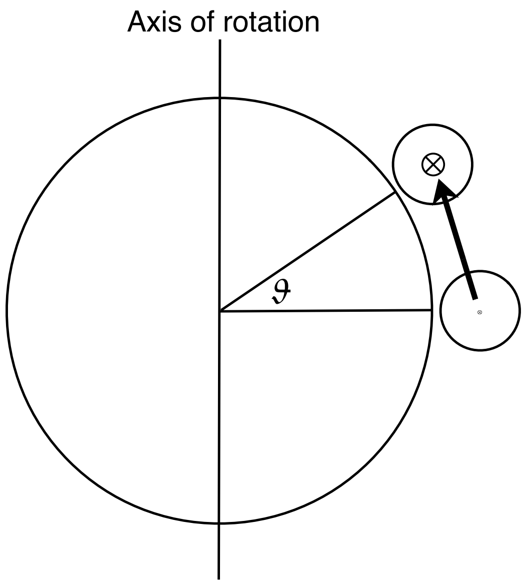


Figure 4.2 If a ring of air at the equator moves polewards it moves closer to the axis of rotation. If the parcels in the ring conserve their angular momentum their zonal velocity must increase; thus, if $m = (\overline{u} + \Omega a \cos\vartheta) a \cos\vartheta$ is preserved and $\overline{u} = 0$ at $\vartheta = 0$ we recover (4.4).

Solution is

$$\overline{u} = \Omega a \frac{\sin^2\vartheta}{\cos\vartheta} \equiv U_M. \tag{4.4}$$

Temperature field

Thermal wind balance:

$$2\Omega \sin\vartheta \frac{\partial u}{\partial z} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta}, \tag{4.5}$$

where $b = g\,\delta\theta/\theta_0$ is the buoyancy and $\delta\theta$ is the deviation of potential temperature from a constant reference value θ_0. (Be reminded that θ is potential temperature, whereas ϑ is latitude.) Vertically integrating from the ground to the height H where the outflow occurs and substituting (4.4) for u yields

$$\frac{1}{a\theta_0}\frac{\partial\theta}{\partial\vartheta} = -\frac{2\Omega^2 a}{gH}\frac{\sin^3\vartheta}{\cos\vartheta}, \tag{4.6}$$

where $\theta = H^{-1}\int_0^H \delta\theta\,\mathrm{d}z$ is the vertically averaged potential temperature. If the latitudinal extent of the Hadley Cell is not too great we can make the small-angle approximation, and replace $\sin\vartheta$ by ϑ and $\cos\vartheta$ by one, then integrating (4.6) gives

$$\theta = \theta(0) - \frac{\theta_0 \Omega^2 y^4}{2gHa^2}, \tag{4.7}$$

where $y = a\vartheta$ and $\theta(0)$ is the potential temperature at the equator, as yet unknown. Away from the equator, the zonal velocity given by (4.4) increases rapidly polewards and the temperature correspondingly drops. How far polewards is this solution valid? And what determines the value of the integration constant $\theta(0)$? To answer these questions we turn to thermodynamics.

4.1.2 Thermodynamics

In the above discussion, the temperature field is slaved to the momentum field in that it seems to follow passively from the dynamics of the momentum equation. Nevertheless, the thermodynamic equation must still be satisfied. Let us assume that the thermodynamic forcing can be represented by a Newtonian cooling to some specified radiative equilibrium temperature, θ_E; this is a severe simplification, especially in equatorial regions where the release of heat by condensation is important. The thermodynamic equation is then

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\theta_E - \theta}{\tau}, \tag{4.8}$$

where τ is a relaxation time scale, perhaps a few weeks. Let us suppose that θ_E falls monotonically from the equator to the pole, and that it increases linearly with height, and a simple representation of this is

$$\frac{\theta_E(\vartheta, z)}{\theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\vartheta) + \Delta_V\left(\frac{z}{H} - \frac{1}{2}\right), \tag{4.9}$$

where Δ_H and Δ_V are non-dimensional constants that determine the fractional temperature difference between the equator and the pole, and the ground and the top of the fluid, respectively. P_2 is the second Legendre polynomial. At $z = H/2$, or for the vertically averaged field, this approximates to

$$\theta_E = \theta_{E0} - \Delta\theta\left(\frac{y}{a}\right)^2, \tag{4.10}$$

where θ_{E0} is the equilibrium temperature at the equator, $\Delta\theta$ determines the equator–pole radiative-equilibrium temperature difference, and

$$\theta_{E0} = \theta_0(1 + \Delta_H/3), \qquad \Delta\theta = \theta_0\Delta_H. \tag{4.11}$$

Now, let us suppose that the solution (4.7) is valid between the equator and a latitude ϑ_H where $v = 0$, so that within this region the system is essentially closed. Conservation of potential temperature then requires that the solution (4.7) must satisfy

$$\int_0^{Y_H} \theta\,\mathrm{d}y = \int_0^{Y_H} \theta_E\,\mathrm{d}y, \tag{4.12}$$

where $Y_H = a\vartheta_H$ is as yet undetermined. Polewards of this, the solution is just $\theta = \theta_E$. Now, we may demand that the solution be continuous at $y = Y_H$ (without temperature continuity the thermal wind would be infinite) and so

$$\theta(Y_H) = \theta_E(Y_H). \tag{4.13}$$

The constraints (4.12) and (4.13) determine the values of the unknowns $\theta(0)$ and Y_H. A little algebra (problem 4.**??**) gives

$$Y_H = \left(\frac{5\Delta\theta g H}{3\Omega^2\theta_0}\right)^{1/2}, \tag{4.14}$$

and

$$\theta(0) = \theta_{E0} - \left(\frac{5\Delta\theta^2 g H}{18a^2\Omega^2\theta_0}\right). \tag{4.15}$$

A useful non-dimensional number that parameterizes these solutions is

$$R \equiv \frac{gH\Delta\theta}{\theta_0\Omega^2a^2} = \frac{gH\Delta_H}{\Omega^2a^2}, \tag{4.16}$$

which is the square of the ratio of the speed of shallow water waves to the rotational velocity of the Earth, multiplied by the fractional temperature difference from equator to pole. Typical values for the Earth's atmosphere are a little less than 0.1. In terms of R we have

$$\boxed{Y_H = a\left(\frac{5}{3}R\right)^{1/2}}, \tag{4.17}$$

and

$$\boxed{\theta(0) = \theta_{E0} - \left(\frac{5}{18}R\right)\Delta\theta}. \tag{4.18}$$

The solution, (4.7) with $\theta(0)$ given by (4.18) is plotted in Fig. 4.3. Perhaps the single most important aspect of the model is that it predicts that the Hadley Cell has a *finite* meridional extent, *even for an atmosphere that is completely zonally symmetric.* The baroclinic instability that does occur in mid-latitudes is not necessary for the Hadley Cell to terminate in the subtropics, although it may be an important factor, or even the determining factor, in the real world.

4.1.3 Zonal wind

The angular-momentum-conserving zonal wind is given by (4.4), which in the small-angle approximation becomes

$$U_M = \Omega\frac{y^2}{a}. \tag{4.19}$$

This relation holds for $y < Y_H$. The zonal wind corresponding to the radiative-equilibrium solution is given using thermal wind balance and (4.10), which leads to

$$U_E = \Omega a R, \tag{4.20}$$

and this holds polewards of Y_H, or ϑ_H, as sketched in Fig. 4.4.

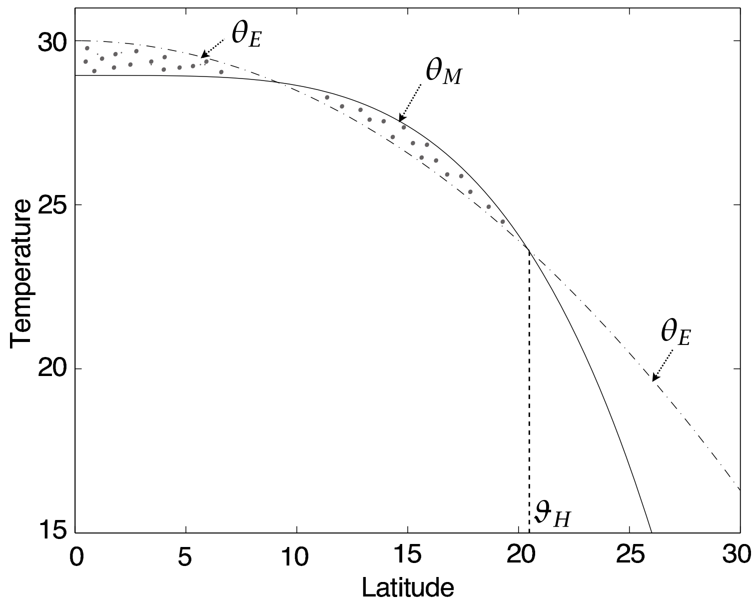


Figure 4.3 The radiative equilibrium temperature (θ_E, dashed line) and the angular-momentum-conserving solution (θ_M, solid line) as a function of latitude. The two dotted regions have equal areas. The parameters are: $\theta_{EO} = 303\,\mathrm{K}$, $\Delta\theta = 50\,\mathrm{K}$, $\theta_0 = 300\,\mathrm{K}$, $\Omega = 7.272 \times 10^{-5}\,\mathrm{s}^{-1}$, $g = 9.81\,\mathrm{m\,s}^{-2}$, $H = 10\,\mathrm{km}$. These give $R = 0.076$ and $Y_H/a = 0.356$, corresponding to $\vartheta_H = 20.4°$.

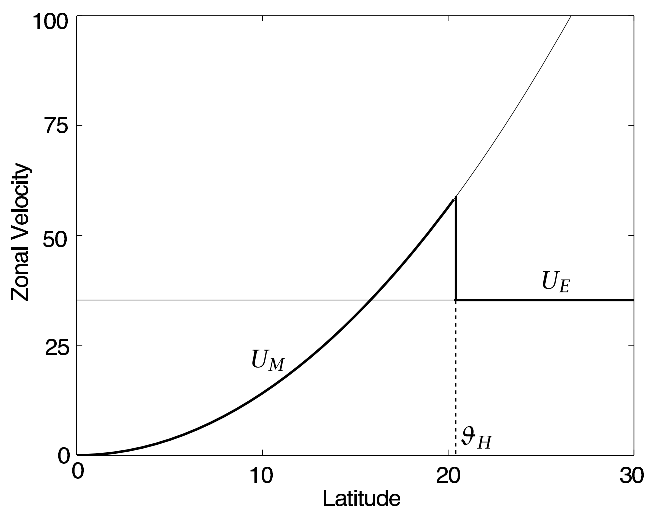


Figure 4.4 The zonal wind corresponding to the radiative equilibrium temperature (U_E,) and the angular-momentum-conserving solution (U_M) as a function of latitude, given (4.19) and (4.20) respectively. The zonal wind (in the model) follows the thick solid line: $u = U_m$ for $\vartheta < \vartheta_H$ ($y < Y_H$), and $u = U_E$ for $\vartheta > \vartheta_H$ ($y > Y_H$), and so has a discontinuity at ϑ_H

4.2 Baroclinic Instability and Termination of the Hadley Cell

One mechanism that could halt the Hadley Cell is baroclinic instability. Having assumed that the surface winds are weak, and knowing the upper level zonal velocity from (4.4), the shear $\partial U_M/\partial z$ is determined by the height of the tropopause H, which we suppose to be a constant. At some latitude ϑ_C the shear will become baroclinically unstable at which point any assumption of zonal symmetry will break down and the Hadley Cell will terminate. What model of baroclinic instability should we use to calculate this? The Eady model has no critical shear — all shears are unstable — but it has no beta-effect and beta is almost certainly important. The Charney model has beta, but it too has no critical shear. However, small shears give rise to shallow, weak instabilities that may not be important. Thus, we are led to the two-level Phillips model of baroclinic instability, because it accounts for the β effect.

In the Phillips model, a flow becomes unstable when it reaches a critical velocity difference between upper and lower levels given by

$$U = U_1 - U_2 = \frac{1}{4}\beta L_d^2 \tag{4.21}$$

where $L_d = NH/f$ is the baroclinic deformation radius, and on the sphere $\beta = 2\Omega\cos\phi/a$. Both β and the f hiding in L_d make this U grow towards the equator and decay towards the pole.

Now, from the Hadley Cell solution

$$U = \Omega a \frac{\sin^2\phi}{\cos\phi} \tag{4.22}$$

so that, modulo constant factors, the Hadley Cell terminates when

$$\frac{\sin^4\phi_c}{\cos^2\phi_c} = \frac{N^2H^2}{\Omega^2a^2}, \tag{4.23}$$

or, with a small angle approximation,

$$\vartheta_H \approx \left(\frac{N^2H^2}{\Omega^2a^2}\right)^{1/4} \sim (NH)^{1/2}. \tag{4.24}$$

As we discussed previously, both theory and modelling suggest that the tropopause will move higher as Global Warming progresses. This model shows that such an increase in H should be accompanied by a poleward expansion of the Hadley cell, perhaps by 1°– 2° over the 21$^{\text{st}}$ century. But perhaps even more significant will be the changes in N^2, which is essentially set by the moist adiabatic lapse rate. A warmer atmosphere will hold more moisture by Clausius-Clapeyron (assuming no major changes in the relative humidity) which reduces the moist adiabatic lapse rate and reduces N^2. Thus the Hadley cell might in fact shrink equator based on this reasoning. However, the value of N in (4.24) should be evaluated at ϑ_H where the baroclinic instability occurs. This is not determined by the moist adiabatic lapse rate, and indeed model results suggest that subtropic static stability may increase with global warming, which would lead to an expansion of the Hadley Cell.

4.3 Effect of Rossby-wave breaking

We will conclude this lecture with an outline of a third model for the extent of the Hadley Cell. Recall we had reduced the zonal momentum equation (4.1) to a balance of two terms; let us now include a third for the momentum balance within the Hadley Cell:

$$(f + \bar{\zeta})\bar{v} = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right). \tag{4.25}$$

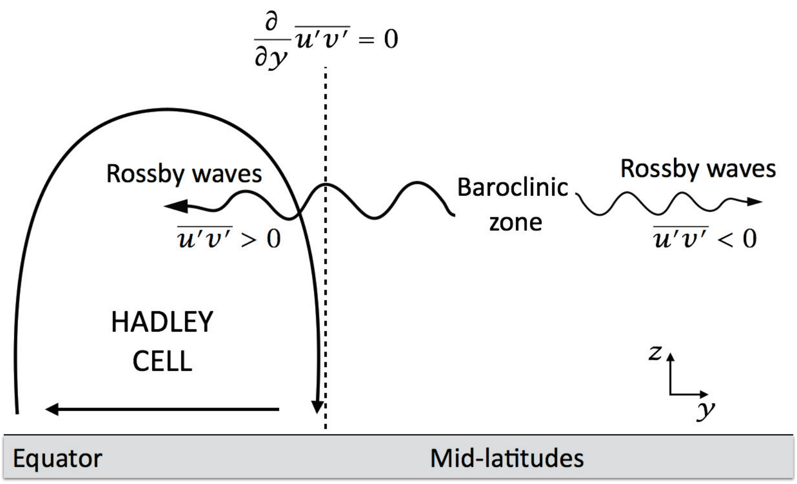


Figure 4.5 A schematic for the mechanism described in section 4.3. Rossby waves are generated through baroclinic instability at mid-latitudes, accelerating the flow eastwards: $\partial(\overline{u'v'})/\partial y > 0$. Some propagate equatorwards, and deposit westward momentum, $\partial(\overline{u'v'})/\partial y < 0$, near the critical latitude inside the Hadley cell. At some latitude the Rossby wave momentum flux is neither convergent nor divergent, $\partial(\overline{u'v'})/\partial y = 0$, corresponding to the edge of the Hadley cell.

However, at the edge of the Hadley Cell we have $\overline{v} = 0$, and thus $\partial_y\left(\overline{u'v'}\right) = 0$. This is not necessarily the latitude where the flow is baroclinically unstable (Section 4.2). Rather, baroclinic instability, occurring at some latitude possibly poleward of here, generates Rossby waves; some of these propagate equatorwards and attenuate as they approach a critical latitude where the mean zonal wind matches the Rossby wave's phase speed (recall the discussion of chapter 3). Recalling our previous discussion, angular momentum conservation initiates a situation with weak winds in low-latitudes and strongly eastward winds in mid-latitudes. Thus a Rossby wave generated at mid-latitude has a phase speed somewhat less than the peak eastward wind speed, but certainly still positive for realistic parameters. This Rossby wave, then, will encounter a critical latitude equatorward of which it cannot flow. The wave breaks near this critical latitude and accelerates the zonal wind westward. This acceleration means that the next Rossby wave will encounter its critical latitude slightly more polewards. We thus have a situation in which the Rossby wave momentum flux convergence $\partial(\overline{u'v'})/\partial y$ is positive in the mid-latitudes and negative in the low-latitudes, requiring a zero crossing $\partial(\overline{u'v'})/\partial y = 0$ at some latitude in between, shown schematically in Figure 4.5. This, as was argued through (4.25), is the edge of the Hadley cell. Note that this edge is equatorward of where the baroclinic instability occurs (which was taken to be the edge in Section 4.2). The precise latitude will be established through a feedback between the eastward acceleration by angular momentum conservation and westward acceleration by Rossby wave breaking.

Chapter 5

The Ocean: Constraints on a Thermal Circulation

We now turn our attention to the ocean. Our goal in this chapter and the next is to describe a theory for the deep circulation of the ocean, sometimes called the meridional overturning circulation (MOC) and occasionally the thermohaline circulation. We begin in this chapter by showing that there have to be winds or some other form of mechanical forcing in order to drive a substantial deep ocean circulation. The root effect goes back to Sandström, and although his rigour was suspect it seems his intuition was right.

5.1 Sandström's Effect

We first give an argument that is similar in spirit to the one that Sandström gave in his original papers (Sandström 1908, 1916).

5.1.1 Maintaining a steady baroclinic circulation

The Boussinesq equations are

$$\frac{D\boldsymbol{v}}{Dt} = -\nabla\phi + b\mathbf{k} + \boldsymbol{F}, \qquad \frac{Db}{Dt} = \dot{Q}, \qquad \nabla\cdot\boldsymbol{v} = 0, \tag{5.1a,b,c}$$

where F represents frictional terms and $\dot{Q} = J + \kappa\nabla^2 b$ (that is, the heating term here includes the effects of diffusion). The circulation, C, changes according to

$$\begin{aligned}\frac{DC}{Dt} = \frac{D}{Dt}\oint \boldsymbol{v}\cdot d\boldsymbol{r} &= \oint\left(\frac{D\boldsymbol{v}}{Dt}\cdot d\boldsymbol{r} + \boldsymbol{v}\cdot d\boldsymbol{v}\right)\\ &= \oint b\mathbf{k}\cdot d\boldsymbol{r} + \oint \boldsymbol{F}\cdot d\boldsymbol{r},\end{aligned} \tag{5.2}$$

(Note that rotation does no work.) Furthermore, we can write rate of change of circulation itself as

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \oint \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) \cdot \mathrm{d}\boldsymbol{r} = \oint \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} \right) \cdot \mathrm{d}\boldsymbol{r}. \tag{5.3}$$

Let us assume the flow is steady, so that $\partial \boldsymbol{v}/\partial t$ vanishes. Let us further choose the path of integration to be a streamline, which since the flow is steady is also a parcel trajectory. The second term on the right-most expression of (5.3) then also vanishes and (5.2) becomes

$$\oint b \, \mathrm{d}z = -\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\oint \frac{\boldsymbol{F}}{|\boldsymbol{v}|} \cdot \boldsymbol{v} \, \mathrm{d}r, \tag{5.4}$$

where the last equality follows because the path is everywhere parallel to the velocity. Let us now assume that the friction retards the flow, and that $\oint \boldsymbol{F} \cdot \boldsymbol{v}/|\boldsymbol{v}| \, \mathrm{d}r < 0$. (One form of friction that has this property is linear drag, $\boldsymbol{F} = -C\boldsymbol{v}$ where C is a constant. The property is similar to, but not the same as, the property that the friction dissipates kinetic energy over the circuit.) Making this assumption, if we integrate the term on the left-hand side by parts we obtain

$$\oint z \, \mathrm{d}b < 0. \tag{5.5}$$

Now, because the integration circuit in (5.6) is a fluid trajectory, the change in buoyancy $\mathrm{d}b$ is proportional to the heating of a fluid element as it travels the circuit $\mathrm{d}b = \dot{Q} \, \mathrm{d}t = \mathrm{d}Q$, where the heating includes diffusive effects).

$$\oint z \, \mathrm{d}Q < 0. \tag{5.6}$$

Thus, the inequality implies that the net heating must be negatively correlated with height: that is, *the heating must occur, on average, at a lower level than the cooling in order that a steady circulation may be maintained against the retarding effects of friction.*

A compressible fluid

A similar result can be obtained for a compressible fluid. We write the baroclinic circulation theorem as

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \oint p \, \mathrm{d}\alpha + \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint T \, \mathrm{d}\eta + \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{5.7}$$

where η is the specific entropy. Then, by precisely the same arguments as led to (5.6), we are led to the requirements that

$$\oint T \, \mathrm{d}\eta > 0 \qquad \text{or equivalently} \qquad \oint p \, \mathrm{d}\alpha > 0. \tag{5.8a,b}$$

Equation (5.8a) means that parcels must gain entropy at high temperatures and lose entropy at low temperatures; similarly, from (5.6b), a parcel must expand ($\mathrm{d}\alpha > 0$) at high pressures and contract at low pressures.

For an ideal gas we can put these statements into a form analogous to (5.6) by noting that $d\eta = c_p(d\theta/\theta)$, where θ is potential temperature, and using the definition of potential temperature, (**??**). With these we have

$$\oint T\, d\eta = \oint c_p \frac{T}{\theta}\, d\theta = \oint c_p \left(\frac{p}{p_R}\right)^{\kappa} d\theta, \tag{5.9}$$

and (5.8a) becomes

$$\boxed{\oint c_p \left(\frac{p}{p_R}\right)^{\kappa} d\theta > 0}\,. \tag{5.10}$$

Because the path of integration is a fluid trajectory, $d\theta$ is proportional to the heating of a fluid element. Thus [and analogous to the Boussinesq result (5.6)], (5.10) implies that *the heating (the potential temperature increase) must occur at a higher pressure than the cooling in order that a steady circulation may be maintained against the retarding effects of friction.*

These results may be understood by noting that the heating must occur at a higher pressure than the cooling in order that work may be done, the work being necessary to convert potential energy into kinetic energy to maintain a circulation against friction. More informally, if the heating is below the cooling, then the heated fluid will expand and become buoyant and rise, and a steady circulation between heat source and heat sink can readily be imagined. On the other hand, if the heating is above the cooling there is no obvious pathway between source and sink.

5.1.2 A rigorous result

Following Paparella & Young (2002) We now show that more rigorously that, if the diffusivity is small, the circulation is in a certain sense weak. Now including molecular viscosity and diffusivity, the equations of motion are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{f} + 2\boldsymbol{\omega}) \times \boldsymbol{v} = -\nabla B + b\mathbf{k} + \nu\nabla^2 \boldsymbol{v}, \tag{5.11a}$$

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \nabla \cdot (b\boldsymbol{v}) = \dot{Q} = J + \kappa\nabla^2 b, \tag{5.11b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{5.11c}$$

Multiply the momentum equation by $\boldsymbol{v}$ and integrate over a volume to give

$$\frac{d}{dt}\left\langle \frac{1}{2}\boldsymbol{v}^2 \right\rangle = \langle wb \rangle - \varepsilon, \tag{5.12}$$

where angle brackets denote a volume average and

$$\varepsilon = -\nu \left\langle \boldsymbol{v} \cdot \nabla^2 \boldsymbol{v} \right\rangle = \nu \left\langle \boldsymbol{\omega}^2 \right\rangle, \tag{5.13}$$

after integrating by parts. The dissipation, ε, is a positive definite quantity.

Write the buoyancy equation as

$$\frac{\mathrm{D}bz}{\mathrm{D}t} = z\frac{\mathrm{D}b}{\mathrm{D}t} + b\frac{\mathrm{D}z}{\mathrm{D}t} = z\dot{Q} + bw, \tag{5.14}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle bz\rangle = \left\langle z\dot{Q}\right\rangle + \langle bw\rangle\,. \tag{5.15}$$

Subtracting (5.15) from (5.12) gives the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{1}{2}\boldsymbol{v}^2 - bz\right\rangle = -\left\langle z\dot{Q}\right\rangle - \varepsilon. \tag{5.16}$$

In a steady state:

$$\left\langle z\dot{Q}\right\rangle = -\varepsilon < 0. \tag{5.17}$$

This is an analogue of our earlier results. It says that if we want to have a dissipative, statistically steady flow there has to be a negative correlation between heating and z. Put simply, the heating has to be below the cooling. But note that the heating and cooling include the diffusive terms.

With diffusion only

Take $\dot{Q} = \kappa\nabla^2 b$ whence (5.17) becomes

$$\frac{\kappa}{V}\int z\nabla^2 b\,\mathrm{d}V = -\varepsilon \tag{5.18}$$

The horizontal part of the integral vanishes so that

$$\frac{\kappa}{H}\int_{-H}^{0} z\frac{\partial^2\overline{b}}{\partial z^2}\,\mathrm{d}z = -\varepsilon. \tag{5.19}$$

where an overbar is a horizontal average. Integrating the LHS by parts gives

$$\frac{\kappa}{H}\left[\overline{b}(0) - \overline{b}(-H)\right] = \varepsilon. \tag{5.20}$$

The LHS is bounded by the surface buoyancy gradient, so the KE dissipation goes to zero as $\kappa \to 0$.

The result is (at least from a physicist's point of view) quite rigorous. It can also be extended to a nonlinear equation of state (Nycander 2010). It tells us there the dissipation of kinetic energy in a fluid diminishes with the diffusivity, and that if $\kappa = 0$ then dissipation vanishes. It doesn't say there is no flow at all, but it is hard to envision a flow in a finite domain that does not dissipated kinetic energy. The result is often characterized as saying that the flow is *non turbulent.*

5.2 Buoyancy and Mixing Driven Scaling Theories

Now we talk about scaling, becoming a bit less rigorous. Interestingly the scaling, dating from Rossby (1965), predates the rigorous theories, and it also provides much stronger bounds. However, it is a scaling and not a rigorous result and therefore open to dispute.

5.2.1 Equations of motion

A non-rotating Boussinesq fluid heated and cooled from above obeys the equations.

$$\frac{D\boldsymbol{v}}{Dt} = -\nabla\phi + \nu\nabla^2\boldsymbol{v} + b\mathbf{k}, \tag{5.21}$$

$$\frac{Db}{Dt} = \kappa\nabla^2 b \tag{5.22}$$

$$\nabla\cdot\boldsymbol{v} = 0. \tag{5.23}$$

with boundary conditions

$$b(x, y, 0, t) = g(x, y), \tag{5.24}$$

For algebraic simplicity consider the two-dimensional version of these equations, in y and z. We can define a streamfunction

$$v = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial y}, \qquad \zeta = \nabla_x^2\psi = \left(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right), \tag{5.25}$$

Taking the curl of the momentum equation gives

$$\frac{\partial\nabla^2\psi}{\partial t} + J(\psi, \nabla^2\psi) = \frac{\partial b}{\partial y} + \nu\nabla^4\psi \tag{5.26a}$$

$$\frac{\partial b}{\partial t} + J(\psi, b) = \kappa\nabla^2 b \tag{5.26b}$$

where $J(a, b) \equiv (\partial_y a)(\partial_z b) - (\partial_z a)(\partial_y b)$.

Non-dimensionalization and scaling

We non-dimensionalize (5.26) by formally setting

$$b = \Delta b\,\hat{b}, \qquad \psi = \Psi\widehat{\psi}, \qquad y = L\hat{y}, \qquad z = H\hat{z}, \qquad t = \frac{LH}{\Psi}\hat{t}, \tag{5.27}$$

where the hatted variables are non-dimensional, Δb is the temperature difference across the surface, L is the horizontal size of the domain, and Ψ, and ultimately the vertical scale H, are to be determined. Substituting (5.27) into (5.26) gives

$$\frac{\partial\widehat{\nabla}^2\widehat{\psi}}{\partial\hat{t}} + \hat{J}(\widehat{\psi}, \nabla^2\widehat{\psi}) = \frac{H^3\Delta b}{\Psi^2}\frac{\partial\hat{b}}{\partial\hat{y}} + \frac{\nu L}{\Psi H}\widehat{\nabla}^4\widehat{\psi}, \tag{5.28a}$$

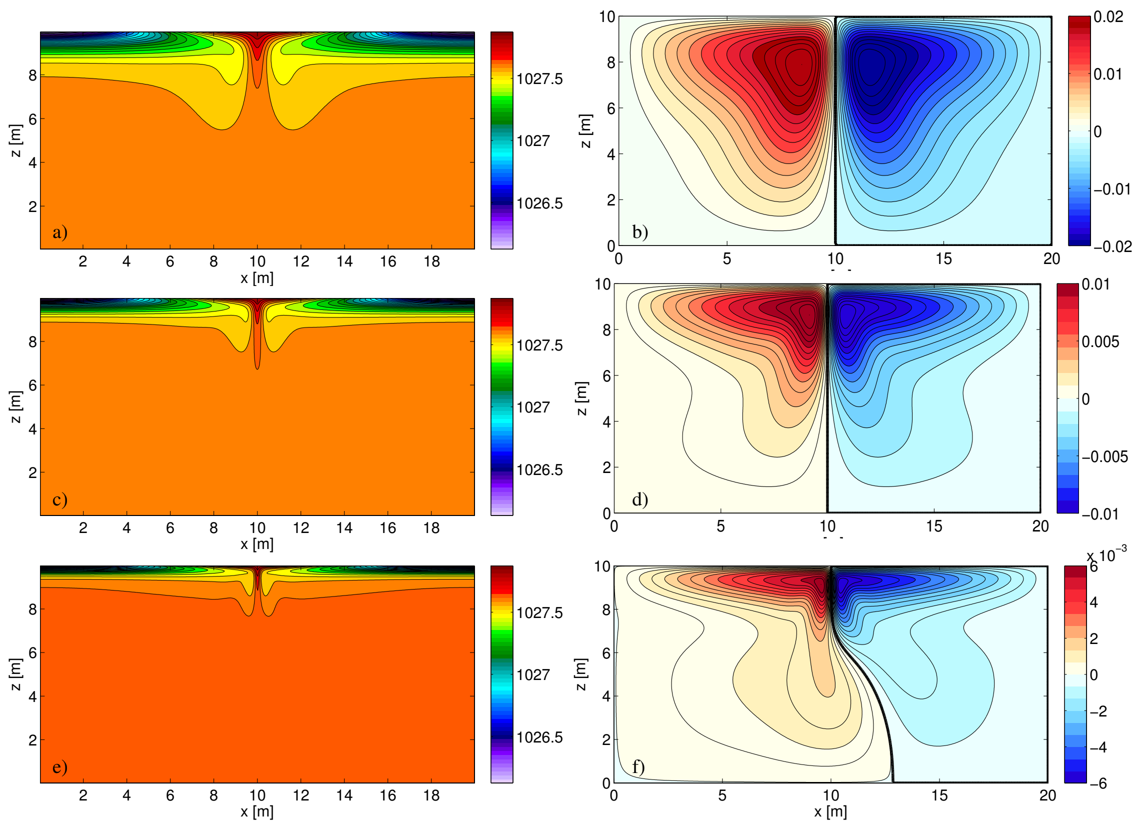


Figure 5.1 Temperature (left) and streamfunction (right) fields. From the top, the Rayleigh numbers are 10^6, 10^7, 10^8.

$$\frac{\partial \hat{b}}{\partial \hat{t}} + \hat{J}(\hat{\psi}, \hat{b}) = \frac{\kappa L}{\Psi H} \widehat{\nabla}^2 \hat{b}, \tag{5.28b}$$

where $\widehat{\nabla}^2 = (H/L)^2 \partial^2/\partial \hat{y}^2 + \partial^2/\partial \hat{z}^2$ and the Jacobian operator is similarly non-dimensional. If we now use (5.28b) to choose Ψ as

$$\Psi = \frac{\kappa L}{H}, \tag{5.29}$$

so that $t = H^2 \hat{t}/\kappa$, then (5.28) becomes

$$\frac{\partial \widehat{\nabla}^2 \hat{\psi}}{\partial \hat{t}} + \hat{J}(\hat{\psi}, \widehat{\nabla}^2 \hat{\psi}) = Ra\sigma\alpha^5 \frac{\partial \hat{b}}{\partial \hat{y}} + \sigma \widehat{\nabla}^4 \hat{\psi}, \tag{5.30}$$

$$\frac{\partial \hat{b}}{\partial \hat{t}} + \hat{J}(\hat{\psi}, \hat{b}) = \widehat{\nabla}^2 \hat{b}, \tag{5.31}$$

and the non-dimensional parameters that govern the behaviour of the system are

$$Ra = \left(\frac{\Delta b L^3}{\nu \kappa} \right), \qquad \text{(the Rayleigh number)}, \tag{5.32a}$$

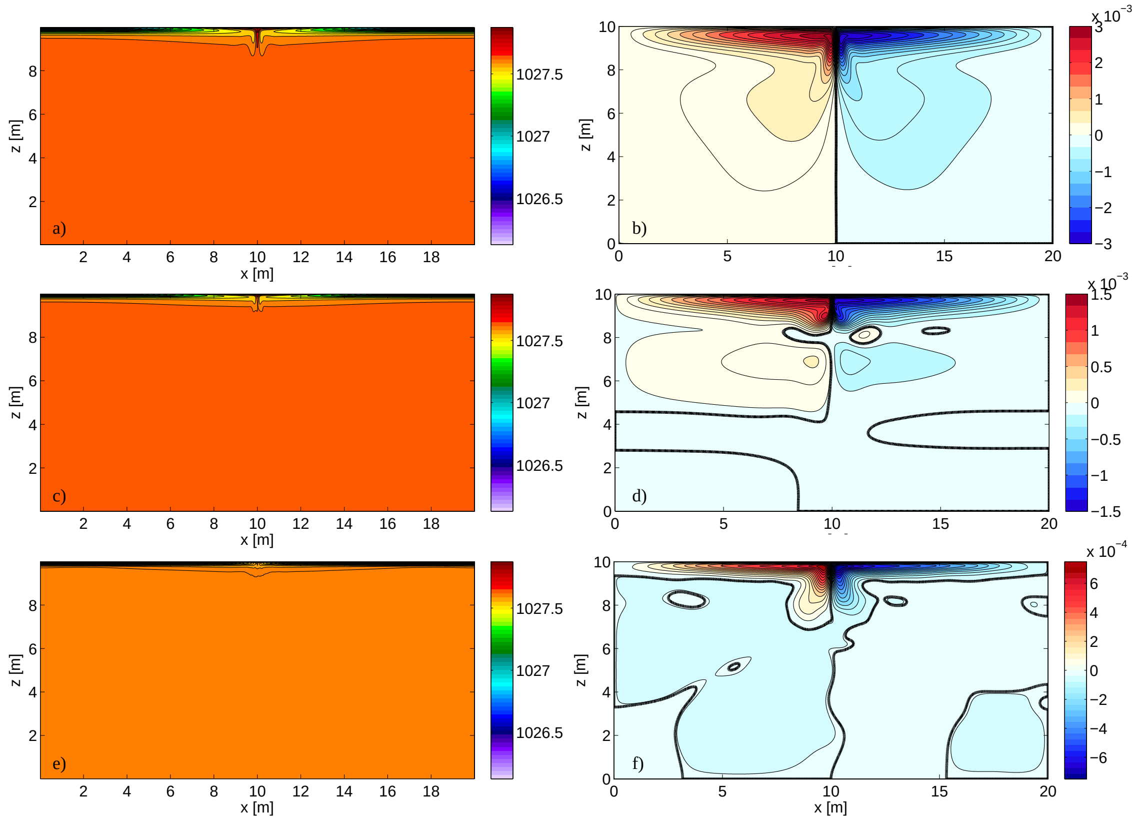


Figure 5.2 Temperature (left) and streamfunction (right) fields. From the top, the Rayleigh numbers are 10^9, 10^{10}, 10^{11}.

$$\sigma = \frac{\nu}{\kappa}, \qquad \text{(the Prandtl number)}, \tag{5.32b}$$

$$\alpha = \frac{H}{L}, \qquad \text{(the aspect ratio)}. \tag{5.32c}$$

5.2.2 Rossby's Scaling

For steady non-turbulent flows, and also perhaps for statistically steady flows, then we can demand that the buoyancy term in (5.30) is $\mathcal{O}(1)$. If it is smaller then the flow is not buoyancy driven, and if it is larger there is nothing to balance it. Our demand can be satisfied only if the vertical scale of the motion adjusts appropriately and, for $\sigma = \mathcal{O}(1)$, this suggests the scalings

$$H = L\sigma^{-1/5}Ra^{-1/5} = \left(\frac{\kappa^2 L^2}{\Delta b}\right)^{1/5}, \qquad \Psi = Ra^{1/5}\sigma^{-4/5}\nu = (\kappa^3 L^3 \Delta b)^{1/5}. \tag{5.33a,b}$$

The vertical scale H arises as a consequence of the analysis, and the vertical size of the domain plays no direct role. [For $\sigma \gg 1$ we might expect the nonlinear terms to be small and if the buoyancy term balances the viscous term in (5.30) the right-hand sides of (5.33) are multiplied by $\sigma^{1/5}$ and $\sigma^{-1/5}$. For seawater, $\sigma \approx 7$ using the molecular values of κ and ν. If small scale turbulence exists, then the eddy viscosity will likely be similar to the eddy diffusivity and $\sigma \approx 1$.] Numerical experiments (Figs. 5.1 and 5.2, taken from Ilicak & Vallis 2012) do provide some support for this scaling, and a few simple and robust points that have relevance to the real ocean emerge, as follows.

- Most of the box fills up with the densest available fluid, with a boundary layer in temperature near the surface required in order to satisfy the top boundary condition. The boundary gets thinner with decreasing diffusivity, consistent with (5.33). This is a diffusive prototype of the oceanic thermocline.
- The horizontal scale of the overturning circulation is large, being at or near the scale of the box.
- The downwelling regions (the regions of convection) are of smaller horizontal scale than the upwelling regions, especially as the Rayleigh number increases.

5.2.3 The importance of mechanical forcing

The above results do not, strictly speaking, prohibit there from being a thermal circulation, with fluid sinking at high latitudes and rising at low, even for zero diffusivity. However, in the absence of any mechanical forcing, this circulation must be laminar, even at high Rayleigh number, meaning that flow is not allowed to break in such a way that energy can be dissipated — a very severe constraint that most flows cannot satisfy. The scalings (5.33) further suggest that the magnitude of the circulation in fact scales (albeit nonlinearly) with the size molecular diffusivity, and if these scalings are correct the circulation will in fact diminish as $\kappa \to 0$. For small diffusivity, the solution most likely to be adopted by the fluid is for the flow to become confined to a very thin layer at the surface, with no abyssal motion at all, which is completely unrealistic vis-à-vis the observed ocean. Thus, the deep circulation of the ocean cannot be considered to be wholly forced by buoyancy gradients at the surface.

Suppose we add a mechanical forcing, $\boldsymbol{F}$, to the right-hand side of the momentum equation (5.11a); this might represent wind forcing at the surface, or tides. The kinetic energy budget becomes

$$\varepsilon = \langle wb \rangle + \langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle = H^{-1}\kappa[\overline{b}(0) - \overline{b}(-H)] + \langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle \,. \tag{5.34}$$

In this case, even for $\kappa = 0$, there is a source of energy and therefore turbulence (i.e., a dissipative circulation) can be maintained. The turbulent motion at small scales then provides a mechanism of mixing and so can effectively generate an 'eddy diffusivity' of buoyancy. *Given* such an eddy diffusivity, wind forcing is no longer necessary for there to be an overturning circulation. Therefore, it is useful to think of mechanical forcing as having two distinct effects.

(i) The wind provides a stress on the surface that may directly drive the large-scale circulation, including the overturning circulation.

(ii) Both tides and the wind provide a mechanical source of energy to the system that allows the flow to become turbulent and so provides a source for an eddy diffusivity and eddy viscosity.

In either case, we may conclude that the presence of mechanical forcing is necessary for there to be an overturning circulation in the world's oceans of the kind observed.

5.3 The Relative Scale of Convective Plumes and Diffusive Upwelling

Why is the downwelling region narrower than the upwelling? The answer is that high Rayleigh number convection is much more efficient than diffusional upwelling, so that the the convective buoyancy flux can match the match the diffusive flux only if the convective plumes cover a much smaller area than diffusion. (Tom Haine explained this to me.) Suppose that the basin is initially filled with water of an intermediate temperature, and that surface boundary conditions of a temperature decreasing linearly from low latitudes to high latitudes are imposed. The deep water will be convectively unstable, and convection at high latitudes (where the surface is coldest) will occur, quickly filling the abyss with dense water. After this initial adjustment, the deep, dense water at lower latitudes will slowly warmed by diffusion, but at the same time surface forcing will maintain a cold high latitude surface, thus leading to high latitude convection. A steady state or statistically steady state is eventually reached with the deep water having a slightly higher potential density than the surface water at the highest latitudes, and so with continual convection, but convection that takes place only at the highest latitudes.

To see this more quantitatively consider the respective efficiencies of the convective heat flux and the diffusive heat flux. Consider an idealized re-arrangement of two parcels, initially with the heavier one on top as illustrated in Fig. 5.3. The potential energy released by the re-arrangement, ΔP is given by

$$\begin{aligned}\Delta P &= P_{\text{final}} - P_{\text{initial}} &(5.35)\\ &= g\left[(\rho_1 z_2 + \rho_2 z_1) - (\rho_1 z_1 + \rho_2 z_2)\right] &(5.36)\\ &= g(z_2 - z_1)(\rho_1 - \rho_2) = \rho_0 \Delta b \Delta z &(5.37)\end{aligned}$$

where $\Delta z = z_2 - z_1$ and $\Delta b = g(\rho_1 - \rho_2)/\rho_0$.

The kinetic energy gained by this re-arrangement, ΔK is given by $\Delta K = \rho_0 w^2$ and equating this to (5.35) gives

$$w^2 = -\Delta b \Delta z. \tag{5.38}$$

Note that if the heavier fluid is initially on top then $\rho_2 > \rho_1$ and, as defined, $\Delta b < 0$. The vertical convective buoyancy flux per unit area, B_c, is given by $B_c = w\Delta b$ and using (5.38) we find

$$B_c = (-\Delta b)^{3/2}(\Delta z)^{1/2}. \tag{5.39}$$

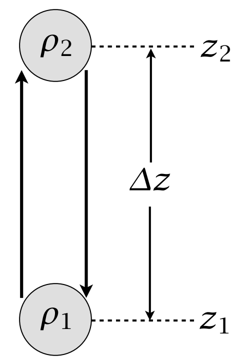


Figure 5.3 Two fluid parcels, of density ρ_1 and ρ_2 and initially at positions z_1 and z_2 respectively, are interchanged. If $\rho_2 > \rho_1$ then the final potential energy is lower than the initial potential energy, with the difference being converted into kinetic energy.

The upwards diffusive flux, B_d, per unit area is given by

$$B_d = \kappa \frac{\Delta b}{H} \tag{5.40}$$

where H the thickness of the layer over which the flux occurs. In a steady state the total diffusive flux must equal the convective flux so that, from (5.39) and (5.40),

$$(-\Delta b)^{3/2}(\Delta z)^{1/2}\delta_A = \kappa \frac{\Delta b}{H}, \tag{5.41}$$

where δ_A is the fractional area over which convection occurs. Thus If we set $\Delta z = H$, we get

$$\delta_A = \frac{\kappa}{(\Delta b)^{1/2} H^{3/2}} \tag{5.42}$$

This is a small number, although it is not quite right yet — we don't really know H. Let us use (5.33a), namely $H = (\kappa^2 L^2/\Delta b)^{1/5}$ then

$$\begin{aligned} \delta_A &= \frac{\kappa}{(\Delta b)^{1/2}(\kappa^2 L^2/\Delta b)^{3/10}} \\ &= \left(\frac{\kappa^2}{\Delta b L^3}\right)^{1/5} = (Ra\,\sigma)^{-1/5}. \end{aligned} \tag{5.43}$$

For geophysically relevant situations this is a very small number, usually smaller than 10^{-5}. Although the details of the above calculation may be questioned (for example, the use of the same buoyancy difference and vertical scale in the convection and the diffusion), the physical basis for the result is clear: for realistic choices of the diffusivity the convection is much more efficient than the diffusion and so will occur over a much smaller area. This result almost certainly transcends the limitations of its derivation.

Chapter 6

The Deep Ocean Circulation

In this chapter we try to understand the processes that give rise to a deep meridional overturning circulation. We'll present a zonally-averaged model of the meridional overturning circulation of the ocean, following Nikurashin & Vallis (2011, 2012). It is a quantitative model, and might even be called a theory, depending on what one's definition of theory is.

6.1 A Model of the Wind-driven Overturning Circulation

The model is motivated by the plot of the stratification shown in Fig. 6.1, and the schematic of water mass properties of the Atlantic shown in Fig. 6.3. The following features are apparent.

1. Two main masses of water, known as North Atlantic Deep Water (NADW) and Antarctic Bottom Water (AABW). Both are interhemispheric. NADW appears to outcrop in high northern latitudes and high southern latitudes, and AABW just at high Southern.
2. Isopycnals are flat over most of the ocean, and slope with a fairly uniform slope in the Southern Ocean.
3. The circulation is along isopycnals in much of the interior. There is some water mass transformation between AABW and NADW, but most of it occurs near the surface.

6.2 A Theory for the MOC in a Single Hemisphere

Let us first imagine there is a wall at the equator, and make a model of the circulation in the Southern Hemisphere; that is, essentially of AABW. The model will have the following features, or bugs if you are being critical.

1. Zonally averaged.
2. Simple geometry. A zonally re-entrant channel at high latitudes, with an enclosed basin between it and the equator.

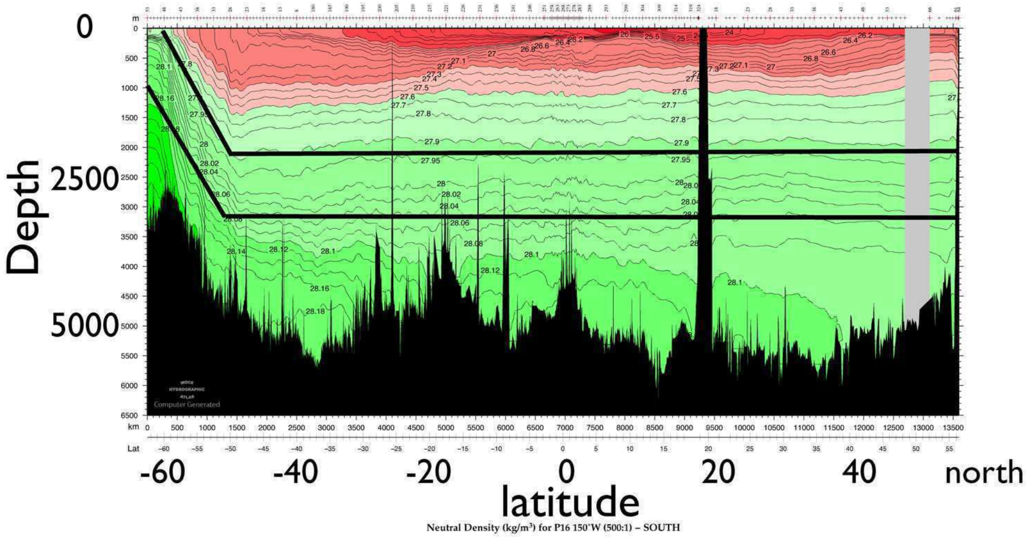


Figure 6.1 Stratification in the Pacific at 150° W

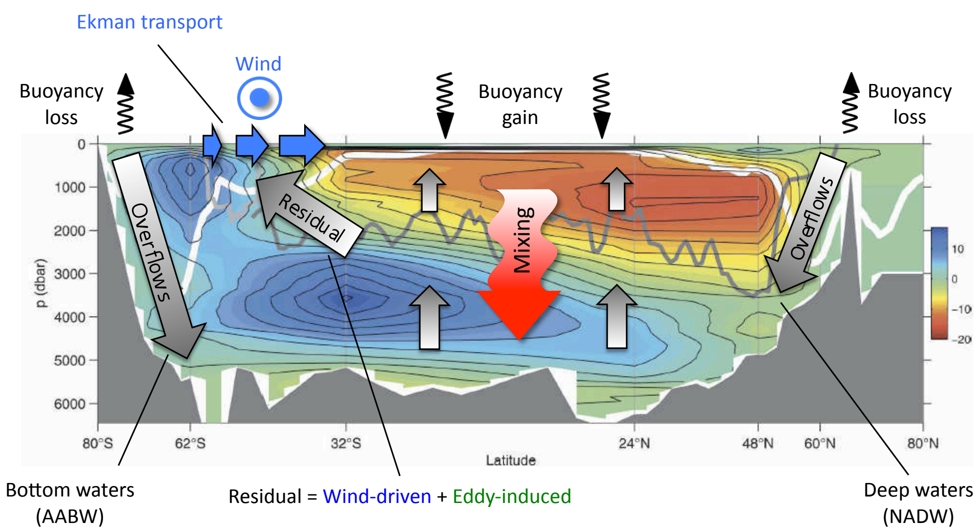


Figure 6.2 A schematic of deep ocean circulation.

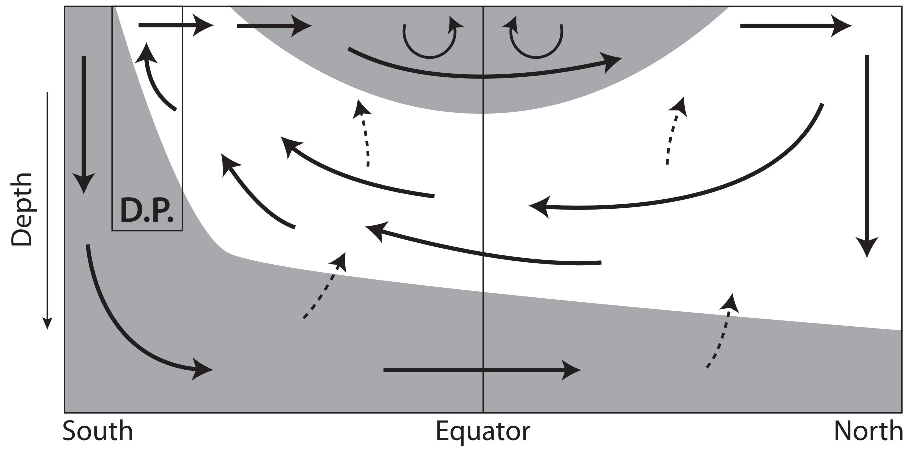


Figure 6.3 A simpler schematic of deep ocean circulation

3. We solve the equations of motion separately in the two regions and match the solutions at the boundary.
4. Mesoscale eddies are parameterized with a very simple down-gradient scheme.
5. There are no wind-driven gyres.

6.2.1 Equations of motion

With quasi-geostrophic scaling the zonally-averaged zonal momentum and buoyancy equations are

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial \tau}{\partial z}, \tag{6.1}$$

$$\frac{\partial \overline{b}}{\partial t} + N^2 \overline{w} = -\frac{\partial}{\partial y}\overline{v'b'} + \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{6.2}$$

where b is buoyancy ('temperature') and $N^2 = \partial_z b_0$. To these we add Thermal wind relation and mass continuity:

$$f_0 \frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \qquad \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0.$$

Define a *residual flow* such that

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right).$$

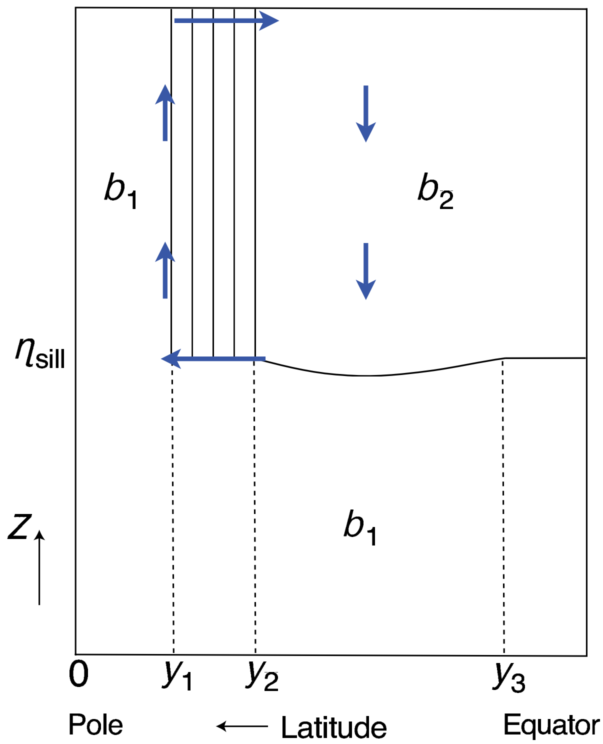


Figure 6.4 Cross-section of a structure of the single-hemisphere ocean model. There is a channel between y_1 and y_2. The arrows indicate the fluid flow driven by the equatorward Ekman transport in the channel, and the solid lines are isopycnals.

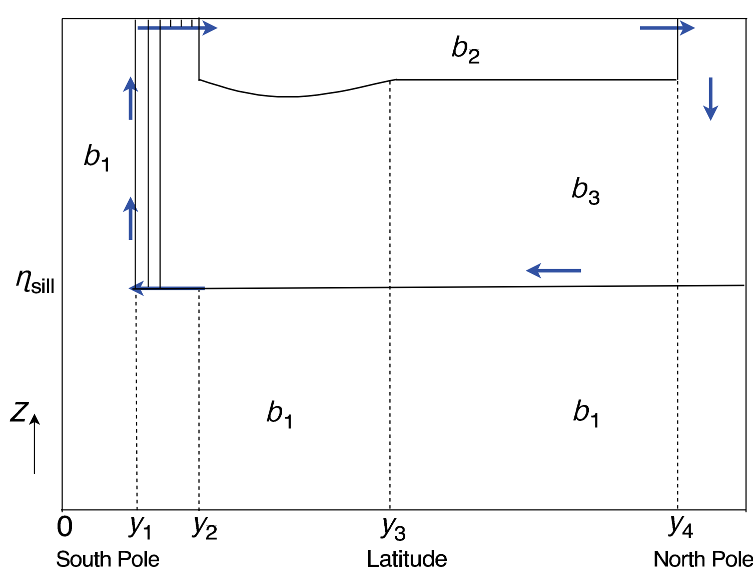


Figure 6.5 As for Fig. 6.4, but now for a two-hemisphere ocean with a source of dense water, b_3, at high northern latitudes.

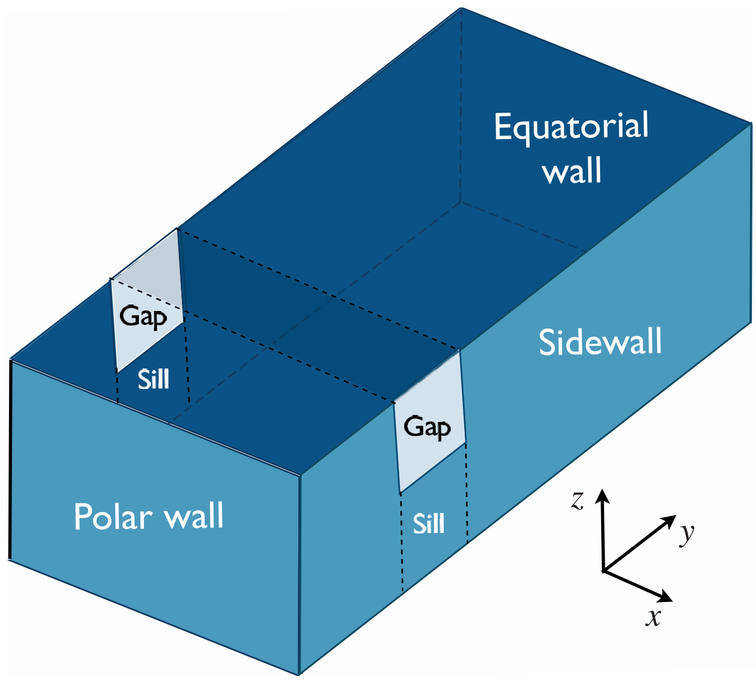


Figure 6.6 Idealized geometry of the Southern Ocean: a re-entrant channel, partially blocked by a sill, is embedded within a closed rectangular basin; thus, the channel has periodic boundary conditions. The channel is a crude model of the Antarctic Circumpolar Current, with the area over the sill analogous to the Drake Passage.

whence

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v}^* = \overline{v'q'} + \frac{\partial \tau}{\partial z} \tag{6.3a}$$

$$\frac{\partial \overline{b}}{\partial t} + N^2 \overline{w}^* = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{6.3b}$$

These are the so-called transformed Eulerian mean (TEM) equations. The theory of them is extensive and suffice it to say here that $\overline{v}^*$ and $\overline{w}^*$ more nearly represent the trajectories of fluid parcels. Note that there are no fluxes in the buoyancy equation and that only the PV flux, $\overline{v'q'}$, need be parameterized. If we now put back some of the terms we omitted, our complete equations are

$$\frac{\partial \overline{u}}{\partial t} - f \overline{v}^* = \overline{v'q'} + \frac{\partial \tau}{\partial z} \tag{6.4a}$$

$$\frac{\partial \overline{b}}{\partial t} + \overline{v}^* \frac{\partial \overline{b}}{\partial y} + \overline{w}^* \frac{\partial \overline{b}}{\partial z} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{6.4b}$$

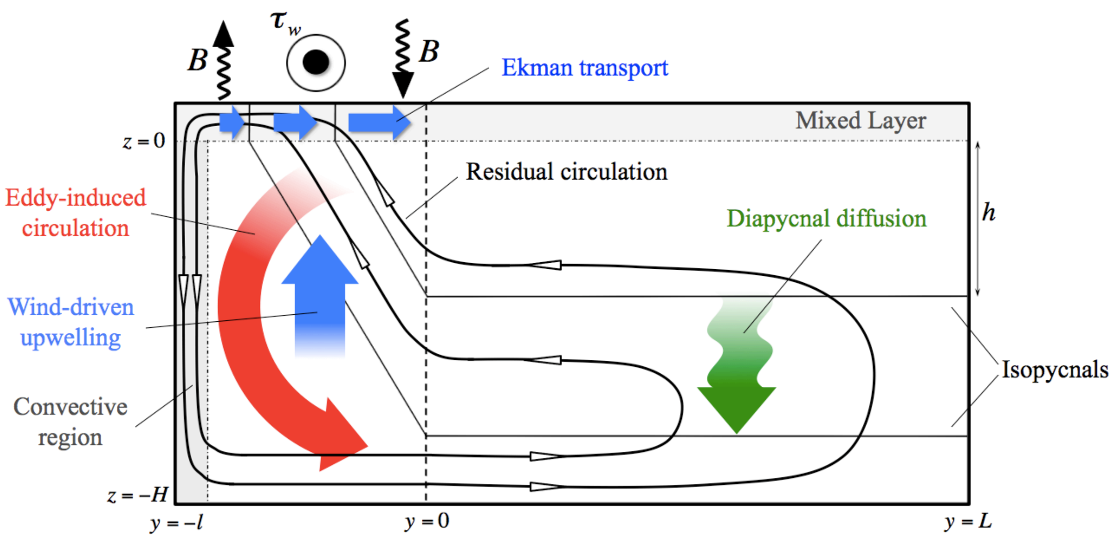


Figure 6.7 Schematic of the single hemisphere meridional overturning circulation crudely representing AABW. Thin black lines are the isopycnals, thick black line is a overturning streamfunction, dashed vertical line is the northern edge of the channel, shaded gray areas are the convective region and the surface mixed layer.

where $(\overline{v}^*, \overline{w}^*) = (-\partial\psi/\partial z, \partial\psi/\partial y)$ and $f\partial\overline{u}/\partial z = -\partial\overline{b}/\partial y$. The stress τ in only non-zero near the top (wind-stress) and bottom (Ekman drag), and τ integrates to zero. We'll look for steady state solutions and drop the * notation so that all variables are residuals and zonal averages.

Equations in the channel

We parameterize

$$\overline{v'q'} = -K_e \frac{\partial \overline{q}}{\partial y}. \tag{6.5}$$

where, approximately, for the large-scale ocean

$$\overline{q} \approx f\frac{\partial}{\partial z}\left(\frac{\overline{b}}{\overline{b}_z}\right), \quad \text{so that} \quad \frac{\partial \overline{q}}{\partial y} \approx f\frac{\partial}{\partial z}\left(\frac{\overline{b}_y}{\overline{b}_z}\right) = -f\frac{\partial S}{\partial z} \tag{6.6}$$

where $S = -\overline{b}_y/\overline{b}_z$ is the slope of the isopycnals. Thus

$$\overline{v'q'} = fK_e \frac{\partial S}{\partial z}. \tag{6.7}$$

and the momentum equation becomes

$$-f\overline{v} = fK_e \frac{\partial S_b}{\partial z} + \frac{\partial \tau}{\partial z}. \tag{6.8}$$

Since $\overline{v}^* = -\partial\psi/\partial z$ we integrate this from the top to a level z and obtain

$$\psi = -\frac{\tau_w}{f} + K_e S. \tag{6.9}$$

We have assumed $\psi = 0$ and $S = 0$ at the top, and $\tau = \tau_w$ at the top (base of mixed layer) and $\tau = 0$ in the interior. [Justify condition on S at the top a bit better.]

The buoyancy equation in terms of streamfunction is

$$\overline{\boldsymbol{v}} \cdot \nabla \overline{b} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2} \qquad \Longrightarrow \qquad \frac{\partial \psi}{\partial y}\frac{\partial \overline{b}}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \overline{b}}{\partial y} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2} \tag{6.10}$$

or

$$\frac{\partial \psi}{\partial y} + S\frac{\partial \psi}{\partial z} = \frac{\partial_z^2 \overline{b}}{\partial_z \overline{b}} . \tag{6.11}$$

The boundary condition on ψ for this will be supplied by the basin! The other boundary condition we will need is the buoyancy distribution at the top, and so we specify

$$\overline{b}(y, z = 0) = b_0(y). \tag{6.12}$$

Equations in the basin

In the basin the slope of the isopycnals is assumed zero and (6.11) becomes the conventional upwelling diffusive balance,

$$w\frac{\partial \overline{b}}{\partial z} = \kappa \frac{\partial^2 \overline{b}}{\partial z^2} \qquad \text{or} \qquad \frac{\partial \psi}{\partial y}\frac{\partial \overline{b}}{\partial z} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2}. \tag{6.13}$$

Integrate this from the edge of the channel, $y = 0$, to the northern edge, $y = L$, and obtain

$$\psi|_{y=0} = -\kappa_v L \frac{\overline{b}_{zz}}{\overline{b}_z} . \tag{6.14}$$

6.2.2 Scaling

Let hats denote non-dimensional values and let

$$z = h\widehat{z}, \qquad y = l\widehat{y}, \qquad \tau_w = \tau_0 \widehat{\tau}_w, \tag{6.15}$$

$$f = f_0 \widehat{f}, \qquad \psi = \frac{\tau_0}{f_0}\widehat{\psi}, \qquad S = \frac{h}{l}\widehat{S}, \tag{6.16}$$

where h is a characteristic vertical scale such that $S \sim h/l$. It will emerge as part of the solution. If we have scaled properly then variables with hats on are of order one. The nondimensional equations of motion are then

$$\text{Buoyancy evolution:} \qquad \partial_{\hat{y}}\hat{\psi} + \hat{S}\partial_{\hat{z}}\hat{\psi} = \epsilon\left(\frac{l}{L}\right)\frac{\partial_{\hat{z}\hat{z}}\hat{b}}{\partial_{\hat{z}}\hat{b}}, \tag{6.17a}$$

$$\text{Momentum balance:} \qquad \hat{\psi} = -\frac{\hat{\tau}}{\hat{f}} + \Lambda\hat{S}, \tag{6.17b}$$

$$\text{Boundary condition:} \qquad \hat{\psi}|_{\hat{y}=0} = -\epsilon\frac{\partial_{\hat{z}\hat{z}}\hat{b}}{\partial_{\hat{z}}\hat{b}}, \tag{6.17c}$$

where

$$\Lambda = \frac{\text{Eddies}}{\text{Wind}} = \frac{K_e}{\tau_0/f_0}\frac{h}{l} \sim 1. \qquad \text{and} \qquad \epsilon = \frac{\text{Mixing}}{\text{Wind}} = \frac{\kappa_v}{\tau_0/f_0}\frac{L}{h} \sim 0.1 - 1. \tag{6.18a,b}$$

These are the two important nondimensional numbers in the problem and we can obtain estimates of their values by using some observed values for the other parameters. Thus, with $\kappa_v = 10^{-5}$ m^2 s^{-1}, $K_e = 10^3$ m^2 s^{-1}, $\tau_0 = 0.1$ N m^{-2}, $f_0 = 10^{-4}$ s^{-1}, $\rho_0 = 10^3$ kg m^{-3}, $L = 10,000$ km, $l = 1,000$ km, and $h = 1$ km we find

$$\Lambda = 1, \qquad \epsilon = 0.1, \qquad \text{and} \qquad \frac{l}{L} = 0.1. \tag{6.19}$$

Note that Λ and ϵ are not independent of each other for they both depend on the vertical scale of stratification h which is a part of the solution, and for that we must look at some limiting cases.

The weak diffusiveness limit

Suppose that mixing is small and that $\epsilon \ll 1$. We can then *require* that $\Lambda = 1$ in order that the eddy-induced circulation nearly balance the wind-driven circulation (because the diffusive term is small), whence the vertical scale h is given by

$$\frac{h}{l} = \frac{\tau_0/f_0}{K_e}. \tag{6.20}$$

If K_e does become small then h becomes large, meaning that the isopycnals are near vertical. Using the above value for h we find that

$$\epsilon = \frac{\kappa_v K_e}{(\tau_0/f_0)^2}\frac{L}{l} \tag{6.21}$$

This is an appropriate measure of the strength of the diapycnal diffusion in the ocean. Using (6.17c) we see that $\widehat{\psi} \sim \epsilon$ so that the dimensional strength of the circulation goes as

$$\Psi = \epsilon \frac{\tau_0}{f_0} = \kappa_v \frac{K_e}{\tau_0/f_0} \frac{L}{l}. \tag{6.22}$$

Another way to obtain this is to note that for weak diffusion the balance in the dimensional momentum equation is between wind forcing and eddy effects (because they must nearly cancel) so that

$$\frac{\tau_w}{f} \sim K_e S, \tag{6.23}$$

which may be written as

$$\frac{h}{l} \sim \frac{\tau_w}{K_e f}. \tag{6.24}$$

Advective-diffusive balance in the basin gives

$$\frac{\partial \psi}{\partial y}\frac{\partial \overline{b}}{\partial z} = \kappa_v \frac{\partial^2 \overline{b}}{\partial z^2} \qquad \Longrightarrow \qquad \Psi = \frac{\kappa_v L}{h} \tag{6.25}$$

and (6.24) and (6.25) together give (6.22).

The strong diffusiveness limit

This limit may be appropriate for the abyssal ocean and in any case it is worth doing, so let us take $\epsilon \gg 1$ and the circulation in the basin will in some sense be strong. As before the nondimensional strength of the circulation is given by

$$\widehat{\psi} = \mathcal{O}(\epsilon) \gg 1. \tag{6.26}$$

The fact that $\widehat{\psi} \neq \mathcal{O}(1)$ means we haven't scaled things in an ideal fashion, but let's proceed anyway. Dimensionally

$$\Psi = \epsilon \frac{\tau_0}{f_0} \qquad \text{or} \qquad \Psi = \frac{\kappa_v L}{h} \tag{6.27a,b}$$

but h and ϵ are both different than before.

Now, if $\widehat{\psi} \sim \epsilon \gg 1$ the diffusion driven circulation in the basin cannot be matched by a purely wind-driven circulation in the channel, since the latter is $\mathcal{O}(1)$. We can only match the circulation with an eddy-driven circulation and therefore we require

$$\Lambda = \mathcal{O}(\epsilon). \tag{6.28}$$

In particular, if we set $\Lambda = \epsilon$ then

$$\epsilon = \Lambda = \sqrt{\left(\frac{K_e \kappa_v L}{(\tau_0/f)^2 l}\right)}. \tag{6.29}$$

This is the square root of the expression for ϵ in the weak diffusiveness limit. Using (6.29) and (6.27a) we find

$$h = \sqrt{\frac{\kappa_v}{K_e} L l} \quad \text{and} \quad \Psi = \sqrt{\frac{K_e \kappa_v L}{l}}. \tag{6.30a,b}$$

Discussion of limits

If diffusion is weak the stratification itself is set by the eddies. Thus, upwelling-diffusion gives $\psi \sim \kappa_v L/h$, with h being set by the wind as in (6.24), thus giving a circulation strength that is linearly proportional to diffusivity, as in (6.22). In the strong diffusion case the diffusion itself affects the stratification, and so we get a weaker dependence of the circulation strength on κ_v. In this limit diapycnal mixing deepens the isopycnals in the basin away from the channel, so that the isopycnals are steeper in the channel. This steepening is balanced the slumping effects of baroclinic instability, and wind only has a secondary effect. From an asymptotic perspective in the small ϵ limit the residual circulation is zero to lowest order. At next order it follows the isopycnals except in the mixed layer.

Instead of varying diffusivity we can think of the wind changing. In the weak wind limit the circulation is diffusively driven and independent of the wind strength. In the strong wind limit the circulation actually *decreases* as the wind increases. This is because the wind steepens the isopycnals so the diffusive term ($\sim \kappa_v \overline{b}_{zz}$) gets smaller and hence the circulation weaker.

6.2.3 Numerical solution of the equations

Matlab code. Show some figures with description.

6.3 An Interhemispheric Circulation

We now introduce another 'water mass' into the mix — *North Atlantic Deep Water*, or NADW. We will construct a model of similar type to what we did in the previous lecture, but now we will divide the ocean into three regions, namely

1. a southern channel, say from 50° S to 70° S
2. a basin region, say from from 50° S to 60° N;
3. a northern convective region.

The idea will be to write down the dynamics in these three regions and match them at the boundaries. The main difference, and it is an important one, between this model and the previous one is the presence of an interhemispheric cell that is primarily wind driven, and sits on top of the lower cell. It is convenient to write down the equations of motion for each region separately, with the first two regions being similar to those of the previous section.

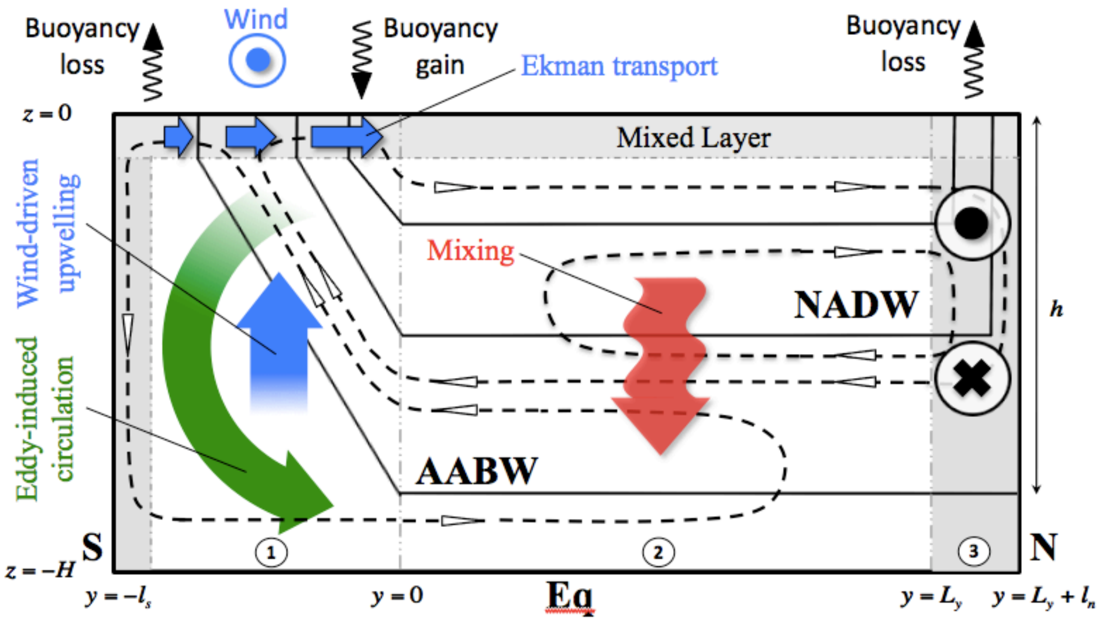


Figure 6.8 Schematic of the interhemispheric MOC. Thin solid black lines are the isopycnals, dashed lines with arrows are the streamlines, dashed vertical lines are the boundaries between adjacent regions, shaded gray areas are the convective regions at high latitudes and the surface mixed layer,and the red arrow represents downward diffusive heat flux. Labels 1, 2, and 3 (in circles) denote the circumpolar channel, ocean basin, and isopycnal outcrop regions.

6.3.1 Equations of motion

In the equations below use restoring conditions at the top, but a specified buoyancy would work too.

Region 1, the southern channel

In the channel the buoyancy equation takes its full advective-diffusive form (although we later find that in some circumstances diffusion is unimportant). The momentum equation has a wind-driven component and and eddy-driven component, as before. In dimensional form the equations are

$$\text{Buoyancy advection:} \qquad J(\psi_1, b_1) = \kappa_v \frac{\partial^2 b_1}{\partial z^2} \tag{6.31a}$$

$$\text{Momentum:} \qquad \psi_1 = -\frac{\tau(y)}{f} - K_e S \tag{6.31b}$$

$$\text{Surface boundary, at } z = 0: \qquad -\kappa \frac{\partial b_1}{\partial z} = \lambda(b^*(y) - b_1) \tag{6.31c}$$

$$\text{Buoyancy match to interior:} \qquad b_1(z)|_{y=0} = b_2(z) \tag{6.31d}$$

$$\text{Streamfunction match to interior} \qquad \psi|_{y=0} = -\kappa_v L \frac{\overline{b}_{zz}}{\overline{b}_z} \tag{6.31e}$$

These are more-or-less the same as those for the channel in the previous section, although we have explicitly added a buoyancy condition and we use a restoring condition on temperature.

Region 2, the basin

In the basin the isopycnals are flat and the buoyancy equation is an upwelling-diffusive balance. We won't need the momentum equation, and so we have

$$\text{Upwelling diffusive:} \qquad \frac{(\psi_3 - \psi_1)}{L} \frac{\partial b_2}{\partial z} = \kappa_v \frac{\partial^2 b_2}{\partial z^2} \tag{6.32a}$$

$$\text{Surface boundary:} \qquad -\kappa \frac{\partial b_2}{\partial z}\bigg|_{z=0} = \lambda(b^* - b_2) \tag{6.32b}$$

The upwelling diffusive balance is just $w\partial_z b = \partial_z^2 b$ with flat isopycnals, with ψ_1 and ψ_3 being the streamfunctions at the southern and northern ends of the basin, respectively. If $\psi_1 \neq \psi_2$ there is a net convergence and hence an upwelling. If $\kappa_v = 0$ then either there is no upwelling or no vertical buoyancy gradient. However, there can be an interhemispheric flow; the properties of the water mass do not change, and we expect the meridional flow to occur in a western boundary current.

Region 3, the northern convective region

In this region the values of buoyancy at the surface (i.e., $b_3(y, z = 0)$) are mapped on to the flat isopycnals in the interior (i.e, $b_2(z)$). We assume this matching occurs by convection. That is, the surface waters convect downward to the level of neutral buoyancy and then move meridionally. By thermal wind the outcropping isopycnals give rise to a zonal flow, with the total zonal transport being determined by the meridional temperature gradient and the depth to which flow convects, which is a function of such things as the winds, eddy strength and diapycnal mixing in the Southern channel. The zonal flow is thus

$$u_3(y, z) = -\frac{1}{f} \int_{-h}^{z} \frac{\partial b_3}{\partial y} \, \mathrm{d}z' + C \tag{6.33}$$

where C is determined by the requirement that $\int_{-h}^{0} u_3 \, \mathrm{d}z = 0$. When the relatively shallow eastward moving zonal flow collides with the eastern wall it subducts and returns. When the deep westward flow collides with the western wall it may move equatorward in a frictional deep western boundary current. It is the upper, northward moving branch of the deep western boundary current that feeds the eastward moving flow. The total volume transport in these zonal flows thus translates to a meridional streamfunction that has the value

$$\psi_3(z) = \int_{-h}^{z} \mathrm{d}z' \int_{L_y}^{L_n} u_3 \,\mathrm{d}y \tag{6.34}$$

where L_y is the latitude of the southern edge of the convecting region and $L_n = L_y + l_n$ is the northern edge of the domain. Summing up, the equations in the convective region are

$$\text{Convective matching:} \qquad b_2(z) \Rightarrow b_3(y, z = 0) \tag{6.35}$$

$$\text{Thermal wind:} \qquad f u_3(y, z) = -\int_{-h}^{z} \frac{\partial b_3}{\partial y} \,\mathrm{d}z' + C \tag{6.36}$$

$$\text{Mass Continuity:} \qquad \psi_3(z) = \iint u_3 \,\mathrm{d}y \,\mathrm{d}z \tag{6.37}$$

We now discuss how all this fits together.

6.3.2 Scaling and Dynamics

Our main focus is on the upper cell, since the lower cell has essentially the same dynamics as in the single hemisphere case. We proceed by writing down some parametric expressions for the streamfunctions in the three domains.

$$\Psi_1 = \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s}\right) L_x, \tag{6.38a}$$

$$\Psi_2 = \Psi_3 - \Psi_1 = \frac{\kappa_v}{h} L_x L_y, \tag{6.38b}$$

$$\Psi_3 = \frac{\Delta b h^2}{f_3}. \tag{6.38c}$$

We don't like these equations because when doing scaling we don't like having additive expressions but for now we damn the torpedoes. The four unknowns are Ψ_1, Ψ_2, Ψ_3 and h and there are four equations (note that (6.38b) is two equations). If we combine them we obtain

$$\frac{\Delta b h^2}{f_3} - \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s}\right) L_x = \frac{\kappa_v}{h} L_x L_y \tag{6.39}$$

This expression is very similar to one obtained by Gnanadesikan (1999).

With no northern source

Suppose that $\Delta b = 0$ and that there is no deep water formation in the North Atlantic. If also κ_v is small then we obtain $h/l_s = (\tau_0/f_1)/K_e$, which is essentially the same as (6.20), obtained previously. If κ_v is large then we find $h^2 = \kappa_v L l_s / K_e$; that is, we recover (6.30a). Pretty much everything is the same as it was section 6.2.

With no southern channel

If there is no southern channel then $\Psi_1 = 0$ and we have

$$\frac{\Delta b h^2}{f_3} = \frac{\kappa_v}{h} L_x L_y \tag{6.40}$$

and

$$h^3 = \kappa_v \left(\frac{f_3 L_x L_y}{\Delta b} \right) \quad \text{and} \quad \Psi_3 = \Psi_2 = (\kappa_v L_x L_y)^{2/3} \left(\frac{\Delta b}{f_3} \right)^{1/3}. \tag{6.41}$$

[Check algebra.] These are classical expressions for the thickness of a diffusive thermocline and the strength of a diffusively-driven overturning circulation, going back to Robinson and Stommel. This is also the same as the strong diffusivity limite.

With all three regions

This is the new bit. The weak diffusivity limit is the interesting case, as the strong diffusivity limit is really just the case with no Southern channel.

In this case the upwelling is weak and $|\Psi_3| \approx \Psi_1|$ and

$$\frac{\Delta b h^2}{f_3} - \left(\frac{\tau_0}{f_1} - K_e \frac{h}{l_s} \right) L_x = 0. \tag{6.42}$$

In this case the basin is just a 'pass-through' region: water formed in the North Atlantic just passes through the basin without change, and upwells in the Southern Ocean. For the moment let us also assume that K_e is small and then

$$\frac{\Delta b h^2}{f_3} = \frac{\tau_0}{\rho_0 f_1} L_x, \tag{6.43}$$

which results in a depth scale h for the stratification,

$$h = \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2} \tag{6.44}$$

Putting in the numbers, we find $h \approx 320$ m. Furthermore, the strength of the circulation is just determined by the wind stress,

$$\Psi_1 = \Psi_3 = \left(\frac{\tau_0 L_x}{f_1} \right) \tag{6.45}$$

which is about 10 Sv.

In the more general case we solve (6.42) to give

$$h = \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b} \right)^{1/2} \left(-\alpha + \sqrt{1 + \alpha^2} \right) \tag{6.46}$$

where α is the nondimensional number given by the ratio of the wind to eddy effects

$$\alpha = \frac{1}{2}\frac{K_e}{l_s}\left(\frac{L_x f_1 f_3}{\tau_0 \Delta b}\right)^{1/2} = \frac{1}{2}\frac{\Psi^*}{\overline{\Psi}}. \tag{6.47}$$

where

$$\overline{\Psi} = \frac{\tau_0}{f_1}L_x, \qquad \Psi^* = -K_e h\frac{L_x}{l_s} = -k_e\frac{L_x}{l_s}\left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b}\right)^{1/2}. \tag{6.48}$$

If we put in numbers then $\alpha \approx 0.08$, $\Psi^* \approx 1.6$ Sv and $\overline{\Psi} \approx 10$ Sv. That is, the wind-induced circulation is the dominant factor in the meridional overturning circulation, and if we take α to small then we have

$$h \approx \left(\frac{\tau_0 f_3 L_x}{f_1 \Delta b}\right)^{1/2}, \qquad \Psi \approx \frac{\tau_0}{f_1}L_x \tag{6.49}$$

Discussion

Although there can be no certainties when eddy diffusivities are present, the use of representative parameters suggests that the eddy-induced circulation is indeed smaller than the wind-driven circulation in the Southern Ocean. That is, putting in numbers, we find $\alpha \approx 0.08$ with $\Psi^* \approx 1.6$ Sv and $\overline{\Psi} \approx 10$ Sv. This suggests that, for typical oceanic parameters, the strength of the eddy-induced circulation on isopycnals corresponding to the middepth overturning cell is only about 10-20% of the wind-driven circulation. Thus, rather than the residual circulation vanishing as is sometimes assumed, the middepth residual circulation is comparable to the wind-driven circulation and acts to pull $O(10)$ Sv of deep water formed at high northern latitudes in the North Atlantic back up to the surface. As a result, the depth scale of stratification h is not linearly proportional to the wind stress τ, as one would obtain from the vanishing residual circulation argument with a simple eddy parameterization, but rather it scales with τ as $\tau^{1/2}$ and is dependent on Δb which is the buoyancy range for isopycnals which are shared between the circumpolar channel and the isopycnal outcrop region in the Northern Hemisphere.

In summary, in the limit of weak diapycnal mixing, relevant to the present middepth ocean, the strength of the middepth overturning circulation is primarily determined by the Ekman transport in the Southern Ocean. The rest of the ocean is essentially forced to adjust and produce the amount of deep water demanded by the Ekman transport and the associated wind-driven upwelling in the Southern Ocean. For instance, during the transient adjustment, the Ekman transport in the circumpolar channel, in conjunction with the surface buoyancy flux, pulls dense waters up from the deep ocean, converts them into light waters at the surface, and pumps these waters into, or just below, the main thermocline in the ocean basin. The rate at which these light waters are then imported into the deep water formation region in the North Atlantic, converted back into dense waters, and exported to the ocean basin at middepth, is controlled by the meridional pressure gradient set up by the outcropping isopycnals in the north. Hence light waters pumped into the ocean basin by the Ekman transport in the south accumulate in, or just below, the main thermocline deepening therefore the middepth isopycnals and increasing the transport of light water into the deep water formation region in the north until the transports in the north and south match. The established interhemispheric balance sets the depth of the isopycnals in the ocean basin and thus stratification throughout the entire ocean.

In the case when deep waters are not produced in the north, as observed in the Pacific Ocean, light waters pumped into the ocean basin by the Southern Ocean wind will deepen the mid-depth isopycnal in the ocean basin, and thus steepen their slopes in the Southern Ocean, until the eddy-induced circulation in the Southern Ocean cancels the wind-driven circulation resulting in a zero residual circulation and water mass transformation.

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