

# Review of the Huang-Hsiung rotating three-dimensional shallow-water equations

by H. Alemi Ardakani & T. J. Bridges

*Department of Mathematics, University of Surrey, Guildford GU2 7XH UK*

— November 22, 2009 —

## 1 Introduction

In the literature there are two strategies for deriving the shallow water equations (SWEs) relative to a rotating frame in three dimensions. The first – the strategy of DILLINGHAM, FALZARANO & PANTAZOPOULOS – is reviewed in the report [3]. The second strategy is that of HUANG & HSIUNG [10, 8, 9, 11] (hereafter HH). In this report the derivation of HH is reviewed identifying the key assumptions.

The HH derivation is then contrasted with a new third strategy for deriving SWEs using the surface equations derived in [2].

## 2 3D equations relative to a moving frame

The starting point is the momentum equations for the fluid in a vessel relative to a body-fixed coordinate system

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + 2\boldsymbol{\Omega} \times \mathbf{u} + \dot{\boldsymbol{\Omega}} \times (\mathbf{x} + \mathbf{d}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\mathbf{x} + \mathbf{d})) + \ddot{\mathbf{q}} + g\mathbf{Q}^T \mathbf{E}_3 = 0, \quad (2.1)$$

where  $g > 0$  is the gravitational constant and

$$\frac{Du}{Dt} := \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}.$$

See Appendix A of [1] for a derivation of the momentum equations relative to a moving frame. In equation (2.1) the angular velocity vector  $\boldsymbol{\Omega}$  and the translational acceleration vector  $\ddot{\mathbf{q}}$  are relative to the body frame.

Conservation of mass takes the usual form

$$u_x + v_y + w_z = 0. \quad (2.2)$$

The boundary conditions at the free surface are the kinematic condition

$$h_t + uh_x + vh_y = w \quad \text{at} \quad z = h(x, y, t), \quad (2.3)$$

and

$$p = 0 \quad \text{at} \quad z = h(x, y, t), \quad (2.4)$$

neglecting surface tension. The boundary condition at the bottom surface and walls is

$$\mathbf{u} \cdot \mathbf{n} = 0. \quad (2.5)$$

Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be a basis for the body frame  $\mathbf{x}$  and  $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$  be a basis for the spatial frame  $\mathbf{X}$ .

### 3 Reduction of conservation of mass

HH start by assuming that  $u$  and  $v$  are functions of horizontal space coordinates and time and do not depend on the vertical space coordinate

$$\begin{aligned} u &:= u(x, y, t), \\ v &:= v(x, y, t). \end{aligned} \quad (\text{HH-1})$$

Then integrating the continuity equation from  $z = 0$  to  $z = h(x, y, t)$  leads to

$$\begin{aligned} \int_0^h (u_x + v_y + w_z) dz &= h(u_x + v_y) + w \Big|_0^h = hu_x + hv_y + h_t + uh_x + vh_y \\ &= h_t + (hu)_x + (hv)_y = 0, \end{aligned} \quad (3.6)$$

using the bottom and kinematic free surface boundary conditions. Note that  $u$  and  $v$  in equation (3.6) can be interpreted as the depth-averaged horizontal velocities,

$$\begin{aligned} \bar{u} &= \frac{1}{h} \int_0^h u(x, y, z, t) dz, \\ \bar{v} &= \frac{1}{h} \int_0^h v(x, y, z, t) dz, \end{aligned} \quad (3.7)$$

since

$$\begin{aligned} (h\bar{u})_x &= \frac{\partial}{\partial x} \int_0^h u(x, y, z, t) dz = h_x u(x, y, z, t) \Big|_0^h + \int_0^h u_x dz \\ (h\bar{v})_y &= \frac{\partial}{\partial y} \int_0^h v(x, y, z, t) dz = h_y v(x, y, z, t) \Big|_0^h + \int_0^h v_y dz \\ \Rightarrow \int_0^h (u_x + v_y + w_z) dz &= (h\bar{u})_x + (h\bar{v})_y - h_x u(x, y, z, t) \Big|_0^h - h_y v(x, y, z, t) \Big|_0^h \\ &\quad + w(x, y, z, t) \Big|_0^h = h_t + (h\bar{u})_x + (h\bar{v})_y = 0, \end{aligned}$$

applying the bottom and kinematic free surface boundary conditions.

### 4 Vertical momentum and the pressure field

The vertical momentum equation can be expressed in the form

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -2(\Omega_1 v - \Omega_2 u) + (\Omega_1^2 + \Omega_2^2)(z + d_3) + \beta(x, y, t), \quad (4.8)$$

where

$$\beta(x, y, t) = -\left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y + d_2) + \left(\dot{\Omega}_2 - \Omega_1\Omega_3\right)(x + d_1) - \ddot{q}_3 - g\mathbf{e}_3 \cdot \mathbf{Q}^T\mathbf{E}_3.$$

The second assumption of HH is to neglect the vertical acceleration

$$\frac{Dw}{Dt} \approx 0. \quad (\text{HH-2})$$

The vertical momentum equation then reduces to

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -2(\Omega_1 v - \Omega_2 u) + (\Omega_1^2 + \Omega_2^2)(z + d_3) + \beta(x, y, t),$$

Integration of this equation from any point  $z$  to the surface  $h$  gives an expression for the pressure field at any point  $z$

$$\begin{aligned} \frac{1}{\rho} p(x, y, z, t) &= (h - z)(2\Omega_1 v - 2\Omega_2 u - \beta(x, y, t)) \\ &\quad + \frac{1}{2}(\Omega_1^2 + \Omega_2^2)((z + d_3)^2 - (h + d_3)^2), \end{aligned} \quad (4.9)$$

where the dynamic free surface boundary condition has been used on the left hand side. In the reduction of the horizontal momentum equations, the horizontal pressure gradient is needed. Differentiating (4.9),

$$\begin{aligned} \frac{1}{\rho} \frac{\partial p}{\partial x} &= (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_x \\ &\quad + \left(2\Omega_1 v_x - 2\Omega_2 u_x - \dot{\Omega}_2 + \Omega_1\Omega_3\right)(h - z). \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \frac{1}{\rho} \frac{\partial p}{\partial y} &= (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_y \\ &\quad + \left(2\Omega_1 v_y - 2\Omega_2 u_y + \dot{\Omega}_1 + \Omega_2\Omega_3\right)(h - z). \end{aligned} \quad (4.11)$$

## 5 Reduction of the horizontal momentum

The  $x$ -component of the momentum equations (2.1) is

$$\begin{aligned} \frac{Du}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= -2(\Omega_2 w - \Omega_3 v) - \dot{\Omega}_2(z + d_3) + \dot{\Omega}_3(y + d_2) \\ &\quad - \Omega_1 \boldsymbol{\Omega} \cdot (\mathbf{x} + \mathbf{d}) + (x + d_1) \|\boldsymbol{\Omega}\|^2 - \ddot{q}_1 - g\mathbf{e}_1 \cdot \mathbf{Q}^T\mathbf{E}_3. \end{aligned} \quad (5.12)$$

Since  $u_z = 0$ ,  $\frac{Du}{Dt}$  reduces to

$$\frac{Du}{Dt} = u_t + uu_x + vu_y.$$

Now substitution of  $\frac{Du}{Dt}$  and  $\frac{1}{\rho} \frac{\partial p}{\partial x}$  from (4.10) into the  $x$ -momentum equation gives

$$\begin{aligned} & u_t + uu_x + vu_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_x \\ & + \left( 2\Omega_1 v_x - 2\Omega_2 u_x - \dot{\Omega}_2 + \Omega_1 \Omega_3 \right) (h - z) + 2(\Omega_2 w - \Omega_3 v) + \dot{\Omega}_2 (z + d_3) \\ & - \dot{\Omega}_3 (y + d_2) + \Omega_1 \boldsymbol{\Omega} \cdot (\mathbf{x} + \mathbf{d}) - (x + d_1) \|\boldsymbol{\Omega}\|^2 + \ddot{q}_1 + g\mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.13)$$

The third assumption of HH is

$$2\Omega_2 w \approx 0. \quad (\text{HH-3})$$

Then integrating equation (5.13) from  $z = 0$  to  $z = h(x, y, t)$  gives

$$\begin{aligned} & u_t + uu_x + vu_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_x \\ & + (\Omega_1 v_x - \Omega_2 u_x + \Omega_1 \Omega_3) h - 2\Omega_3 v + \left( \Omega_1 \Omega_2 - \dot{\Omega}_3 \right) (y + d_2) \\ & - (\Omega_2^2 + \Omega_3^2)(x + d_1) + \left( \dot{\Omega}_2 + \Omega_1 \Omega_3 \right) d_3 + \ddot{q}_1 + g\mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.14)$$

The fourth assumption of HH is

$$(\Omega_1 v_x - \Omega_2 u_x) h \approx 0. \quad (\text{HH-4})$$

Note that  $\Omega_1 v_x - \Omega_2 u_x$  is half of the  $x$ -derivative of vertical component of the Coriolis force. Finally equation (5.14) simplifies to

$$\begin{aligned} & u_t + uu_x + vu_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_x \\ & + (h + d_3) \Omega_1 \Omega_3 - 2\Omega_3 v + \left( \Omega_1 \Omega_2 - \dot{\Omega}_3 \right) (y + d_2) \\ & - (\Omega_2^2 + \Omega_3^2)(x + d_1) + \dot{\Omega}_2 d_3 + \ddot{q}_1 + g\mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.15)$$

This is the form of the  $x$ -momentum component of the SWEs that appears in [10, 11].

## 5.1 Reduction of the $y$ -momentum equation

The  $y$ -component of the momentum equations (2.1) is

$$\begin{aligned} \frac{Dv}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial y} &= 2(\Omega_1 w - \Omega_3 u) + \dot{\Omega}_1 (z + d_3) - \dot{\Omega}_3 (x + d_1) \\ &\quad - \Omega_2 \boldsymbol{\Omega} \cdot (\mathbf{x} + \mathbf{d}) + (y + d_2) \|\boldsymbol{\Omega}\|^2 - \ddot{q}_2 - g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3. \end{aligned} \quad (5.16)$$

Since  $v_z = 0$ ,  $\frac{Dv}{Dt}$  reduces to

$$\frac{Dv}{Dt} = v_t + uv_x + vv_y.$$

Now substitution of  $\frac{Dv}{Dt}$  and  $\frac{1}{\rho} \frac{\partial p}{\partial y}$  from (4.11) into the  $y$ -momentum equation gives

$$\begin{aligned} & v_t + uv_x + vv_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_y \\ & + \left( 2\Omega_1 v_y - 2\Omega_2 u_y + \dot{\Omega}_1 + \Omega_2 \Omega_3 \right) (h - z) - 2(\Omega_1 w - \Omega_3 u) - \dot{\Omega}_1 (z + d_3) \\ & + \dot{\Omega}_3 (x + d_1) + \Omega_2 \boldsymbol{\Omega} \cdot (\mathbf{x} + \mathbf{d}) - (y + d_2) \|\boldsymbol{\Omega}\|^2 + \ddot{q}_2 + g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.17)$$

The fifth assumption is

$$2\Omega_1 w \approx 0. \quad (\text{HH-5})$$

Then integrating equation (5.17) from  $z = 0$  to  $z = h(x, y, t)$  gives

$$\begin{aligned} & v_t + uv_x + vv_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_y \\ & + (\Omega_1 v_y - \Omega_2 u_y + \Omega_2 \Omega_3) h + 2\Omega_3 u + \left( \Omega_1 \Omega_2 + \dot{\Omega}_3 \right) (x + d_1) \\ & - (\Omega_1^2 + \Omega_2^2)(y + d_2) + \left( -\dot{\Omega}_1 + \Omega_2 \Omega_3 \right) d_3 + \ddot{q}_2 + g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.18)$$

The sixth assumption is

$$(\Omega_1 v_y - \Omega_2 u_y) h \approx 0, \quad (\text{HH-6})$$

note that  $\Omega_1 v_y - \Omega_2 u_y$  is half of the  $y$ -derivative of vertical component of the Coriolis force. Finally equation (5.18) simplifies to

$$\begin{aligned} & v_t + uv_x + vv_y + (2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t)) h_y \\ & + (h + d_3) \Omega_2 \Omega_3 + 2\Omega_3 u + \left( \Omega_1 \Omega_2 + \dot{\Omega}_3 \right) (x + d_1) \\ & - (\Omega_1^2 + \Omega_2^2)(y + d_2) - \dot{\Omega}_1 d_3 + \ddot{q}_2 + g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 = 0. \end{aligned} \quad (5.19)$$

This is the form of the  $y$ -momentum component of the SWEs that appears in [10, 11].

The assumptions (HH-4) and (HH-6) can be written in a concise vector form

$$h \nabla [(\boldsymbol{\Omega} \times \mathbf{u}) \cdot \mathbf{e}_3] \approx 0.$$

## 6 Prescribed yaw-pitch-roll motion of the vessel

The rotation matrix in [9, 10, 11] is restricted to

$$\mathbf{Q} = \begin{bmatrix} \cos \psi \cos \theta & -\sin \psi \cos \phi + \cos \psi \sin \theta \sin \phi & \boxed{\sin \psi} \sin \phi + \cos \psi \sin \theta \cos \phi \\ \sin \psi \cos \theta & \cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi & -\cos \psi \sin \phi + \sin \psi \sin \theta \boxed{\cos \phi} \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix}. \quad (\text{HH-7})$$

This matrix is equation (7) in [11] with two typos corrected. The first boxed term is incorrectly typed as  $\sin \theta$  in [11] and the second boxed term is incorrectly typed

as  $\sin \phi$  in [11]. The correspondence with the notation in [11] is  $\phi = e_1$ ,  $\theta = e_2$  and  $\psi = e_3$ .

The connection between the rotation matrix (**HH-7**) and a yaw-pitch-roll sequence is best demonstrated using rotation tensors, and the details of this construction are given in the report [6].

The angular velocity can be deduced using the matrix representation or by using the properties of rotation tensors. The spatial angular velocity is

$$\boldsymbol{\Omega}^s = \dot{\psi} \mathbf{E}_3 + \dot{\theta} \mathbf{a}_2 + \dot{\phi} \mathbf{b}_1.$$

The 3-2-1 Euler angles have a singularity at  $\theta = \pm \frac{1}{2} \pi$  and so  $\theta$  must be restricted to the interval

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

The singularity arises because the map from  $(\dot{\psi}, \dot{\theta}, \dot{\phi})$  to  $\boldsymbol{\Omega}^s$  is not invertible when  $\cos \theta = 0$ . To see this write out the map

$$\begin{pmatrix} \Omega_1^s \\ \Omega_2^s \\ \Omega_3^s \end{pmatrix} = \begin{bmatrix} 0 & -\sin \psi & \cos \psi \cos \theta \\ 0 & \cos \psi & \sin \psi \cos \theta \\ 1 & 0 & -\sin \theta \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}.$$

The determinant of the coefficient matrix is  $-\cos \theta$  resulting in a singularity when  $\cos \theta = 0$ .

The body angular velocity is

$$\begin{aligned} \boldsymbol{\Omega}^b &= \dot{\psi}(-\mathbf{E}_1 \sin \theta + \mathbf{E}_2 \cos \theta \sin \phi + \mathbf{E}_3 \cos \theta \cos \phi) \\ &\quad + \dot{\theta}(\mathbf{E}_2 \cos \phi - \mathbf{E}_3 \sin \phi) + \dot{\phi} \mathbf{E}_1 \\ &= \mathbf{E}_1(\dot{\phi} - \dot{\psi} \sin \theta) + \mathbf{E}_2(\dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi) \\ &\quad + \mathbf{E}_3(\dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi). \end{aligned}$$

In [14] the body angular velocity is denoted by  $\boldsymbol{\omega}_{0\mathbf{R}}$  and the above expression for  $\boldsymbol{\Omega}^b$  agrees with  $\boldsymbol{\omega}_{0\mathbf{R}}$  on page 188 of [14]. In matrix form

$$\begin{pmatrix} \Omega_1^b \\ \Omega_2^b \\ \Omega_3^b \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}.$$

This mapping is also singular when  $\cos \theta = 0$ . Write this mapping in matrix-vector form

$$\boldsymbol{\Omega}^b = \mathbf{B}^{-1} \dot{\Theta}.$$

Then

$$\mathbf{B} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix}.$$

As shown below in §6.1, this expression agrees with  $[B] := \mathbf{B}$  in [7] and in [11].

Further details on the 3–2–1 Euler angles are in the report [6].

## 6.1 Notation in [11]

On page 222 of [11] the ship translational displacements, denoted by  $(\xi_1, \xi_2, \xi_3)$ , in the steady moving system and the Eulerian angles, denoted by  $(e_1, e_2, e_3)$ , are related by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{bmatrix} [R] & 0 \\ 0 & [B] \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}. \quad (6.20)$$

This is equation (5) in [11]. In equations (6)-(7) in [11], the matrices  $[B]$  and  $[R]$  are defined as follows

$$[B] = \begin{bmatrix} 1 & \sin e_1 \tan e_2 & \cos e_1 \tan e_2 \\ 0 & \cos e_1 & -\sin e_1 \\ 0 & \sin e_1 \sec e_2 & \cos e_1 \sec e_2 \end{bmatrix},$$

and

$$[R] = \begin{bmatrix} \cos e_2 \cos e_3 & \sin e_1 \sin e_2 \cos e_3 - \cos e_1 \sin e_3 & \cos e_1 \sin e_2 \cos e_3 + \sin e_1 \boxed{\sin e_2} \\ \cos e_2 \sin e_3 & \sin e_1 \sin e_2 \sin e_3 + \cos e_1 \cos e_3 & \boxed{\sin e_1} \sin e_2 \sin e_3 - \sin e_1 \cos e_3 \\ -\sin e_2 & \sin e_1 \cos e_2 & \cos e_1 \cos e_2 \end{bmatrix}.$$

Replacing  $e_1 = \phi$ ,  $e_2 = \theta$ ,  $e_3 = \psi$  and  $\xi_1 = q_1$ ,  $\xi_2 = q_2$  and  $\xi_3 = q_3$  it is clear (after correcting two boxed typos in  $[R]$ ) that  $[R] = \mathbf{Q}$  and  $[B] = \mathbf{B}$ . The first boxed term should be  $\sin e_2 \rightarrow \sin e_3$  and the second boxed term should be  $\sin e_1 \rightarrow \cos e_1$ . In (6.20), the angular velocity  $(u_4, u_5, u_6)$  is related to the Euler angles by

$$\begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = [B]^{-1} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix}.$$

Hence it is clear that  $u_4 = \Omega_1^b$ ,  $u_5 = \Omega_2^b$ ,  $u_6 = \Omega_3^b$ .

## 7 Final form of the HH-SWEs

The rotation matrix that HH have used in their derivation is (**HH-7**). So

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 &= -\sin \theta, \\ \mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 &= \cos \theta \sin \phi, \\ \mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3 &= \cos \theta \cos \phi, \end{aligned}$$

Table 1: Correspondence between notation in [2] and notation in [10, 11]

Ref [2]	HH
$d_1$	$-x_g$
$d_2$	$-y_g$
$d_3$	$-z_g$
$\ddot{q}_1$	$\dot{u}_1$
$\ddot{q}_2$	$\dot{u}_2$
$\ddot{q}_3$	$\dot{u}_3$
$\phi$	$e_1$
$\theta$	$e_2$
$\psi$	$e_3$
$\Omega_1$	$u_4$
$\Omega_2$	$u_5$
$\Omega_3$	$u_6$

and so the  $x$ -momentum equation becomes

$$\begin{aligned}
 & u_t + uu_x + vu_y + (2\Omega_1v - 2\Omega_2u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t))h_x \\
 & + (h + d_3)\Omega_1\Omega_3 - 2\Omega_3v + (\Omega_1\Omega_2 - \dot{\Omega}_3)(y + d_2) \\
 & - (\Omega_2^2 + \Omega_3^2)(x + d_1) + \dot{\Omega}_2d_3 + \ddot{q}_1 - g \sin \theta = 0.
 \end{aligned} \tag{7.21}$$

In HH notation this equation is written

$$u_t + uu_x + vu_y + (-q_2 - q_3h)h_x = q_0 + q_1h,$$

with  $q_0, q_1, q_2, q_3$  defined in [10]. The  $y$ -momentum equation becomes

$$\begin{aligned}
 & v_t + uv_x + vv_y + (2\Omega_1v - 2\Omega_2u - (\Omega_1^2 + \Omega_2^2)(h + d_3) - \beta(x, y, t))h_y \\
 & + (h + d_3)\Omega_2\Omega_3 + 2\Omega_3u + (\Omega_1\Omega_2 + \dot{\Omega}_3)(x + d_1) \\
 & - (\Omega_1^2 + \Omega_3^2)(y + d_2) - \dot{\Omega}_1d_3 + \ddot{q}_2 + g \cos \theta \sin \phi = 0.
 \end{aligned} \tag{7.22}$$

In HH notation this equation is written [10]

$$v_t + uv_x + vv_y + (-r_2 - r_3h)h_y = r_0 + r_1h.$$

with  $r_2 = q_2$ ,  $r_3 = q_3$  and  $r_0, r_1$  defined in [10]. The function  $\beta(x, y, t)$  becomes

$$\beta(x, y, t) = -(\dot{\Omega}_1 + \Omega_2\Omega_3)(y + d_2) + (\dot{\Omega}_2 - \Omega_1\Omega_3)(x + d_1) - \ddot{q}_3 - g \cos \theta \cos \phi.$$

Unwrapping the flux vector form of equation (55) in [10] shows that they are equivalent to (3.6), (7.21) and (7.22) here.

The identification of notation is as shown in Table 1.



## 8 Correction of typographical errors

There are some typos in [10]. The terms in boxes below are ones that have been corrected. Equation (44) in [10] is

$$v_t + uv_x + vv_y + wv_z = f_2 - \frac{1}{\rho} \frac{\partial p}{\partial y} - \dot{u}_2 - P_2 - Q_2 - \boxed{R_2} - S_2,$$

and equation (45) is

$$w_t + uw_x + vw_y + ww_z = \boxed{f_3} - \frac{1}{\rho} \frac{\partial p}{\partial z} - \dot{u}_3 - P_3 - Q_3 - R_3 - S_3,$$

and (46) is

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \boxed{+} g \sin(e_2) \\ -g \sin(e_1) \cos(e_2) \\ -g \cos(e_1) \cos(e_2) \end{bmatrix},$$

and  $\vec{f}_0$  in (57) is

$$\vec{f}_0 = \begin{bmatrix} 0 \\ f_{02} \\ \boxed{0} \end{bmatrix},$$

and equation (59) is

$$\begin{aligned} q_0 = & \boxed{+} g \sin(e_2) - \dot{u}_1 + 2u_6v + (u_5^2 + u_6^2)(x - x_g) \\ & (\dot{u}_6 - u_4u_5)(y - y_g) + (u_4u_6 \boxed{+} \dot{u}_5) z_g, \end{aligned}$$

and equation (61) is

$$\begin{aligned} q_2 = & -g \cos(e_1) \cos(e_2) - \dot{u}_3 - 2(u_4v - u_5u) - (u_4u_6 - \dot{u}_5)(x - x_g) \\ & - (u_5u_6 \boxed{+} \dot{u}_4)(y - y_g) - (u_4^2 + u_5^2) z_g, \end{aligned}$$

and  $r_0$  in (63) is

$$\begin{aligned} r_0 = & -g \sin(e_1) \boxed{\cos(e_2)} - \dot{u}_2 - 2u_6u - (u_4u_5 + \dot{u}_6)(x - x_g) \\ & + (u_4^2 + u_6^2)(y - y_g) + (u_5u_6 - \dot{u}_4) z_g. \end{aligned}$$

## 9 Comparison of the HH SWEs with the SWEs in [2]

The new surface equations for rotating 3D shallow water flow derived in [2] are

$$\begin{aligned} U_t + UU_x + VU_y + a_{11} h_x + a_{12} h_y &= b_1, \\ V_t + UV_x + VV_y + a_{21} h_x + a_{22} h_y &= b_2, \end{aligned} \tag{9.1}$$

with

$$\begin{aligned}
a_{11} &= 2\Omega_1 V + \mathbf{e}_3 \cdot \mathbf{Q}^T \ddot{\mathbf{q}} + g\mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3 - (\Omega_1^2 + \Omega_2^2)(h + d_3) \\
&\quad - (\dot{\Omega}_2 - \Omega_1 \Omega_3)(x + d_1) + (\dot{\Omega}_1 + \Omega_3 \Omega_2)(y + d_2), \\
a_{22} &= -2\Omega_2 U - (\Omega_1^2 + \Omega_2^2)(h + d_3) + \mathbf{e}_3 \cdot \mathbf{Q}^T \ddot{\mathbf{q}} + g\mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3 \\
&\quad + (\dot{\Omega}_1 + \Omega_2 \Omega_3)(y + d_2) - (\dot{\Omega}_2 - \Omega_1 \Omega_3)(x + d_1).
\end{aligned} \tag{9.2}$$

and

$$\begin{aligned}
a_{12} &= 2\Omega_2 V, \\
a_{21} &= -2\Omega_1 U,
\end{aligned} \tag{9.3}$$

and

$$\begin{aligned}
b_1 &= -2\Omega_2 h_t + 2\Omega_3 V - \mathbf{e}_1 \cdot \mathbf{Q}^T \ddot{\mathbf{q}} - g\mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 + (\Omega_2^2 + \Omega_3^2)(x + d_1) \\
&\quad + (\dot{\Omega}_3 - \Omega_1 \Omega_2)(y + d_2) - (\dot{\Omega}_2 + \Omega_1 \Omega_3)(h + d_3), \\
b_2 &= 2\Omega_1 h_t - 2\Omega_3 U - \mathbf{e}_2 \cdot \mathbf{Q}^T \ddot{\mathbf{q}} - g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 + (\Omega_1^2 + \Omega_3^2)(y + d_2) \\
&\quad - (\dot{\Omega}_3 + \Omega_1 \Omega_2)(x + d_1) + (\dot{\Omega}_1 - \Omega_2 \Omega_3)(h + d_3).
\end{aligned} \tag{9.4}$$

In equation (9.1) the translational acceleration vector  $\ddot{\mathbf{q}}$  is relative to the spatial frame and if we replace  $\mathbf{Q}^T \ddot{\mathbf{q}}$  with body translational acceleration vector  $\ddot{\mathbf{q}}$  then the coefficients simplify to

$$\begin{aligned}
a_{11} &= 2\Omega_1 V + \ddot{q}_3 + g\mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3 - (\Omega_1^2 + \Omega_2^2)(h + d_3) \\
&\quad - (\dot{\Omega}_2 - \Omega_1 \Omega_3)(x + d_1) + (\dot{\Omega}_1 + \Omega_3 \Omega_2)(y + d_2), \\
a_{22} &= -2\Omega_2 U - (\Omega_1^2 + \Omega_2^2)(h + d_3) + \ddot{q}_3 + g\mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3 \\
&\quad + (\dot{\Omega}_1 + \Omega_2 \Omega_3)(y + d_2) - (\dot{\Omega}_2 - \Omega_1 \Omega_3)(x + d_1),
\end{aligned} \tag{9.5}$$

and

$$\begin{aligned}
b_1 &= -2\Omega_2 h_t + 2\Omega_3 V - \ddot{q}_1 - g\mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3 + (\Omega_2^2 + \Omega_3^2)(x + d_1) \\
&\quad + (\dot{\Omega}_3 - \Omega_1 \Omega_2)(y + d_2) - (\dot{\Omega}_2 + \Omega_1 \Omega_3)(h + d_3), \\
b_2 &= 2\Omega_1 h_t - 2\Omega_3 U - \ddot{q}_2 - g\mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3 + (\Omega_1^2 + \Omega_3^2)(y + d_2) \\
&\quad - (\dot{\Omega}_3 + \Omega_1 \Omega_2)(x + d_1) + (\dot{\Omega}_1 - \Omega_2 \Omega_3)(h + d_3).
\end{aligned} \tag{9.6}$$

To compare the HH SWEs with new surface equations write the HH momentum equations in the following form

$$\begin{aligned}
u_t + uu_x + vu_y + a_{11}^{HH} h_x &= b_1^{HH}, \\
v_t + uv_x + vv_y + a_{22}^{HH} h_y &= b_2^{HH},
\end{aligned} \tag{9.7}$$

where

$$\begin{aligned}
a_{11}^{HH} &= 2\Omega_1 v - 2\Omega_2 u - (\Omega_1^2 + \Omega_2^2)(h + d_3) + (\dot{\Omega}_1 + \Omega_2 \Omega_3)(y + d_2) \\
&\quad - (\dot{\Omega}_2 - \Omega_1 \Omega_3)(x + d_1) + \ddot{q}_3 + g\mathbf{e}_3 \cdot \mathbf{Q}^T \mathbf{E}_3, \\
a_{22}^{HH} &= a_{11}^{HH},
\end{aligned}$$

and

$$\begin{aligned}
b_1^{HH} &= -(h + d_3) \Omega_1 \Omega_3 + 2\Omega_3 v - \left( \Omega_1 \Omega_2 - \dot{\Omega}_3 \right) (y + d_2) \\
&\quad + (\Omega_2^2 + \Omega_3^2) (x + d_1) - \dot{\Omega}_2 d_3 - \ddot{q}_1 - g \mathbf{e}_1 \cdot \mathbf{Q}^T \mathbf{E}_3, \\
b_2^{HH} &= -(h + d_3) \Omega_2 \Omega_3 - 2\Omega_3 u - \left( \Omega_1 \Omega_2 + \dot{\Omega}_3 \right) (x + d_1) \\
&\quad + (\Omega_1^2 + \Omega_3^2) (y + d_2) + \dot{\Omega}_1 d_3 - \ddot{q}_2 - g \mathbf{e}_2 \cdot \mathbf{Q}^T \mathbf{E}_3.
\end{aligned}$$

Assume that the  $(U, V)$  velocity fields in the surface equations are equivalent to the velocity fields in the HH equations, and then a comparison of coefficients shows that

$$\begin{aligned}
a_{11}^{HH} &= a_{11} - 2\Omega_2 u, \\
a_{22}^{HH} &= a_{22} + 2\Omega_1 v, \\
a_{12}^{HH} &= a_{12} - 2\Omega_2 v, \\
a_{21}^{HH} &= a_{21} + 2\Omega_1 u, \\
b_1^{HH} &= b_1 + 2\Omega_2 h_t + \dot{\Omega}_2 h, \\
b_2^{HH} &= b_2 - 2\Omega_1 h_t - \dot{\Omega}_1 h.
\end{aligned} \tag{9.8}$$

Hence, assuming equivalence of the horizontal velocity fields, the HH SWEs reduce to the surface equations when

$$|2\Omega_2 u| \ll 1, \quad |2\Omega_1 v| \ll 1, \quad |2\Omega_2 v| \ll 1, \quad |2\Omega_1 u| \ll 1,$$

and

$$|2\Omega_2 h_t + \dot{\Omega}_2 h| \ll 1, \quad |2\Omega_1 h_t + \dot{\Omega}_1 h| \ll 1.$$

## 10 Numerical simulations

Numerical simulations are performed using the new surface SWEs in [2]. The numerical scheme is detailed in [2, 5]. In the simulations the initial conditions are

$$U(x, y, 0) = V(x, y, 0) = 0 \quad \text{and} \quad h(x, y, 0) = h_0,$$

with  $h_0$  input. Other required inputs are

$$L_1, \quad L_2, \quad d_1, \quad d_2, \quad d_3,$$

and the functions associated with the rigid body motion of the vessel

$$\Omega_1, \quad \Omega_2, \quad \Omega_3, \quad q_1, \quad q_2, \quad q_3.$$

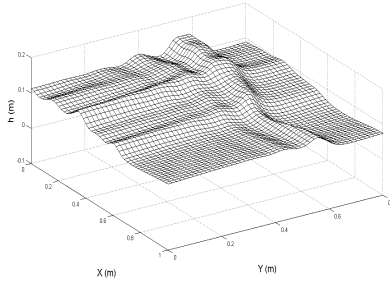


Figure 1: Wave pattern due to surge and sway: comparison of the numerics based on the surface SWEs with Figure 26 of [10]. The data are  $t = 1.5 \text{ sec}$ ,  $\omega = 6.0 \text{ rad/sec}$ ,  $L_1 = 1 \text{ m}$  and  $L_2 = 0.8 \text{ m}$ .

## 10.1 Numerical results for surge-sway forcing

This example is motivated by the numerics in [10]. They set the following parameter values

$$L_1 = 1, \quad L_2 = 0.8, \quad h_0 = 0.10, \quad d_1 = -0.50, \quad d_2 = -0.40, \quad d_3 = 0.0,$$

with all units in metres. There is no rotation so  $\Omega_1 = \Omega_2 = \Omega_3 = 0$ . The translational motion is surge and sway and they take the form

$$q_1(t) = q_2(t) = \varepsilon_5 \sin(\omega_4 t), \quad q_3(t) = 0,$$

with

$$\varepsilon_4 = 0.02, \quad \varepsilon_5 = 0.02, \quad \text{and} \quad \omega_4 = 6.0,$$

with the amplitude in metres and the units of  $\omega_4$  are radians per second.

A example simulation using the code in [2, 5] is shown in Figure 1. The wave surface due to the coupled surge and sway motions is shown at  $t = 1.50 \text{ sec}$ . The corresponding velocity distribution is shown in Figure 2. Further discussion of these results is in [2].

## 10.2 Numerical results for yaw forcing

In this section numerical results for pure yaw motion are shown. The parameter values are the same as the surge-sway example with a different forcing. The forcing is pure yaw motion so that  $q_1 = q_2 = q_3 = 0$  and

$$\mathbf{Q} = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the angular velocity is

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = \dot{\psi}.$$

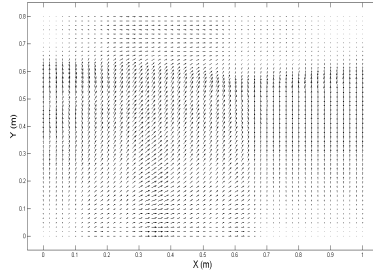


Figure 2: Velocity field due to surge and sway: comparison of the numerics based on the surface SWEs with Figure 27 of [10]. The data are  $t = 1.5 \text{ sec}$ ,  $\omega = 6.0 \text{ rad/sec}$ ,  $L_1 = 1 \text{ m}$  and  $L_2 = 0.8 \text{ m}$ .

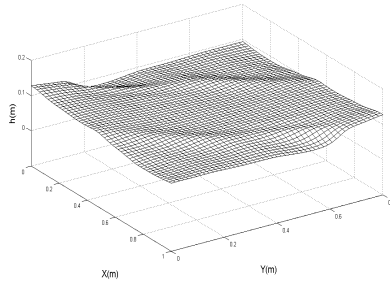


Figure 3: Wave pattern due to yaw: comparison of the numerics based on the surface SWEs with Figure 28 of [10]. The data are  $t = 1.5 \text{ sec}$ ,  $\omega = 6.0 \text{ rad/sec}$ ,  $L_1 = 1 \text{ m}$  and  $L_2 = 0.8 \text{ m}$ .

The yaw angle is prescribed to be harmonic

$$\psi(t) = \varepsilon_3 \sin(\omega_3 t),$$

with

$$\varepsilon_3 = 4.0^\circ = \frac{\pi}{45} \text{ rad}, \quad \text{and} \quad \omega_3 = 6.0 \text{ rad/sec}.$$

The wave surface under only the yaw motion at  $t = 1.50 \text{ sec}$  is depicted in figure 3. The corresponding velocity distribution is shown in figure 4. Further discussion of these results is in [2].

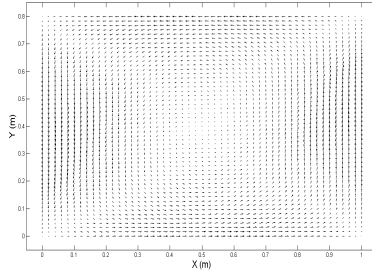


Figure 4: Velocity field due to yaw: comparison of the numerics based on the surface SWEs with Figure 29 of [10]. The data are  $t = 1.5 \text{ sec}$ ,  $\omega = 6.0 \text{ rad/sec}$ ,  $L_1 = 1 \text{ m}$  and  $L_2 = 0.8 \text{ m}$ .

### 10.3 Some properties of the SWEs for pure yaw forcing

In the case of pure yaw forcing, the coefficients in the surface SWEs reduce considerably. With  $\Omega_1 = \Omega_2 = 0$  and  $\Omega_3 = \dot{\psi}$ ,

$$\begin{aligned} a_{11} &= a_{22} = g \\ a_{12} &= a_{21} = 0 \\ b_1 &= 2\Omega_3 V + \Omega_3^2(x + d_1) + \dot{\Omega}_3(y + d_2) \\ b_2 &= -2\Omega_3 U + \Omega_3^2(y + d_2) - \dot{\Omega}_3(x + d_1). \end{aligned}$$

In this case the surface equations and the HH equations are identical and the momentum equations reduce to forced classical SWEs

$$\begin{aligned} U_t + UU_x + VU_y + gh_x &= 2\dot{\psi}V + \dot{\psi}^2(x + d_1) + \ddot{\psi}(y + d_2) \\ V_t + UV_x + VV_y + gh_y &= -2\dot{\psi}U + \dot{\psi}^2(y + d_2) - \ddot{\psi}(x + d_1). \end{aligned} \quad (10.9)$$

These equations have the remarkable property that they conserve a form of potential vorticity. Let

$$\mathcal{P} := \frac{V_x - U_y + 2\dot{\psi}}{h}.$$

It is proved in [2] that  $\mathcal{P}$  is a Lagrangian invariant

$$\frac{D\mathcal{P}}{Dt} = 0.$$

## References

- [1] H. ALEMI ARDAKANI & T.J. BRIDGES. *Shallow-water sloshing in rotating vessels. Part 1. Two-dimensional flow field*, Preprint: University of Surrey (2009).

- [2] H. ALEMI ARDAKANI & T.J. BRIDGES. *Shallow-water sloshing in rotating vessels. Part 2. Three-dimensional flow field*, Preprint: University of Surrey (2009).
- [3] H. ALEMI ARDAKANI & T.J. BRIDGES. *Review of the Dillingham, Falzarano & Pantazopoulos rotating three-dimensional shallow-water equations*, Internal Report: University of Surrey (2009).
- [4] H. ALEMI ARDAKANI & T.J. BRIDGES. *Review of the Huang-Hsiung rotating two-dimensional shallow-water equations*, Internal Report: University of Surrey (2009).
- [5] H. ALEMI ARDAKANI & T.J. BRIDGES. *Shallow water sloshing in rotating vessels: details of the numerical algorithm*, Internal Report: University of Surrey (2009).
- [6] H. ALEMI ARDAKANI & T.J. BRIDGES. *Review of the 3–2–1 Euler angles: the yaw–pitch–roll sequence*, Internal Report: University of Surrey (2009).
- [7] Z. HUANG. *Nonlinear shallow water flow on deck and its effect on ship motion*, PhD Thesis, Technical University of Nova Scotia (1995).
- [8] Z. HUANG & C.C. HSIUNG. *Application of the flux difference splitting method to compute nonlinear shallow water flow on deck*, Proceedings of ninth international workshop on water waves and floating bodies, Japan (1994).
- [9] Z. HUANG & C.C. HSIUNG. *Nonlinear shallow water flow on a three-dimensional deck*, Proceedings of tenth international workshop on water waves and floating bodies, England (1995).
- [10] Z. HUANG & C.C. HSIUNG. *Nonlinear shallow-water flow on deck*, J. Ship Research **40** 303–315 (1996).
- [11] Z. HUANG & C.C. HSIUNG. *Nonlinear shallow-water flow on deck coupled with ship motion*, Proceedings of the twenty-first symposium on naval hydrodynamics (1997).
- [12] C. LEUBNER. *Coordinate-free rotation operator*, Amer. J. Phys. **47** 727–729 (1979).
- [13] C. LEUBNER. *Correcting a widespread error concerning the angular velocity of a rotating rigid body*, Amer. J. Phys. **49** 232–234 (1981).
- [14] O.M. O'REILLY. *Intermediate Dynamics for Engineers: a Unified Treatment of Newton-Euler and Lagrangian Mechanics*, Cambridge University Press (2008).