Normal-form Based Analysis of Climate Time Series

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Outline

- Time series analysis
- Saddle-node induced tipping
- Estimate of normal form parameters from time series
Tipping in palaeoclimate time series

- End of last glaciation
- End of younger Dryas

Temperature vs. time (yrs) for the end of last glaciation and end of younger Dryas.
Background — time series analysis

**Experiment**

\[
\begin{align*}
\text{input} & \quad u \\
\text{output} & \quad y
\end{align*}
\]

\[
\begin{align*}
u & \quad \uparrow \quad u \\
y & \quad \downarrow \quad y
\end{align*}
\]

**Model**

\[
\begin{align*}
\dot{x}(t) & = f(t, x, p) \\
x(t_0) & = x_0
\end{align*}
\]
Background — time series analysis

Experiment

Model

\[ y(t) = f(t, x, p) \]
\[ x(t_0) = x_0 \]

Observation
Background — time series analysis

Low-dim chaos & small noise

High-dim chaos or large noise

Embedding, recurrence

Assumption

- quasi-stationary

Questions

- finite $t$
- qualitative change of attractor

[Kantz & Schreiber, 2000]
Background — time series analysis

**Low-dim chaos & small noise**

- embedding, recurrence

- [Kantz & Schreiber, 2000]

**High-dim chaos or large noise**

- Assumption
  - quasi-stationary

- Questions
  - finite $t$
  - qualitative change of attractor
Tipping — Mechanical caricature of positive feedback

- squishy beam, clamped and loaded with gradually increasing mass $m$
Tipping — Mechanical caricature of positive feedback

saddle-node normal form with drift and noise

\[
\frac{dx}{dt} = -V'(x) = \mu - x^2 + \text{noise}
\]

\[
\frac{d\mu}{dt} = -\epsilon
\]

\[
\mu \sim m_{\text{critical}} - m
\]
Tipping — Mechanical caricature of positive feedback

\[ V(\chi) \]

\[ \mu \sim 1 \]

\[ \mu \approx 0 \]

\[ \mu > 0 \]

\[ \mu < 0 \]
Estimate from time series

Approach to **Saddle-node**  \( \dot{x} = f(x, \mu) + \sigma \eta_t \)

parameter  \( \dot{\mu} = -\epsilon \)

stable  \( x_{eq} \)

unstable
Estimate from time series

Approach to **Saddle-node**

\[ \dot{x} = f(x, \mu) + \sigma \eta_t \]

approx \( x_{eq} \)

parameter \( \dot{\mu} = -\epsilon \)

stable \( x_{eq} \)

unstable

zero order

\[ \dot{x} = -\epsilon [x - x_{eq}(\epsilon t)] + ? \]
Approach to **Saddle-node** $\dot{x} = f(x, \mu) + \sigma \eta_t$

- **approx** $x_{eq}$
- **stable** $x_{eq}$
- **unstable**

- **zero order**
  $$\dot{x} = -(\kappa)[x - x_{eq}(\epsilon t)] + ?$$

- **first order**
  $$\dot{x} = -\kappa(\epsilon t)[x - x_{eq}(\epsilon t)] + ?$$

- **fit**
- **AR(1)**
- **DFA**
- **Var**

**Estimate from time series**
Estimate of linear decay rate

First order
\[ \dot{x} = -\kappa(\epsilon t)x + \sigma \eta_t \]

AR(1) (Held & Kleinen’04)
DFA (Livina & Lenton’07)

Variance
Estimate of linear decay rate

\[ \dot{x}(t) = -\kappa(\epsilon t)x + \sigma \eta_t \]

first order

AR(1) (Held&Kleinen’04)
DFA (Livina&Lenton’07)

Variance

AR(1) \( x(t_{n+1}) = \alpha x(t_n) \) \( \Rightarrow \) fit \( \alpha \) \( \Rightarrow \) \( \alpha = \exp(-\kappa \Delta t) \)
Estimate of linear decay rate

\[ \dot{x} = -\kappa(\epsilon t)x + \sigma \eta_t \]

**AR(1)** (Held & Kleinen '04)

DFA (Livina & Lenton '07)

**Variance**

\[ x(t_{n+1}) = \alpha x(t_n) \rightarrow \text{fit } \alpha \rightarrow \alpha = \exp(-\kappa \Delta t) \]

\[ \text{Var} = \frac{\sigma^2}{\kappa} \]

stationary distribution normal
When linear is not enough

\[ \kappa \] and \( d \) need to know

\[ \mu \not\sim \varepsilon t \]

Assuming saddle-node

\[ \kappa = 0 \]

Escape prob.

equation

\[ \chi \]

\[ d \]

\[ \mu \not\sim \varepsilon t \]

Need to know \( \kappa \) and \( d \)
Back to estimates

\[ \frac{\dot{x}}{x} = -\kappa \epsilon t \]

\[ \sigma \eta_t \]

noise

\[ \text{AR(1)} \] (Held & Kleinen’04)

DFA (Livina & Lenton’07)

Variance

\[ \text{AR(1)} \ x(t_{n+1}) = \alpha x(t_n) \Rightarrow \text{fit } \alpha \Rightarrow \alpha = \exp(-\kappa \Delta t) \]

\[ \text{Variance} \]

\[ \text{Var} = \frac{\sigma^2}{\kappa} \]

stationary distribution normal
Back to estimates

\[
\frac{\dot{x}}{x} = -\kappa(\epsilon t)x + N x^2 + \sigma \eta_t
\]

**AR(1)** (Held & Kleinen’04)

DFA (Livina & Lenton’07)

**Variance**

\[
\text{Var} = \frac{\sigma^2}{\kappa}
\]

stationary distribution normal

**AR(1)**  \( x(t_{n+1}) = \alpha x(t_n) \)  \( \Rightarrow \)  fit \( \alpha \)  \( \Rightarrow \)  \( \alpha = \exp(-\kappa \Delta t) \)
Back to estimates

\[ \dot{x} = -\kappa(\epsilon_t)x + N x^2 + \sigma \eta_t \]

\textbf{AR(1)} (Held\&Kleinen’04)

DFA (Livina\&Lenton’07)

\textbf{Variance}

\[ \text{Var} = \frac{\sigma^2}{\kappa} \]

\text{stationary distribution normal}
Back to estimates

\[ \dot{x} = -\kappa(\epsilon t)x + N x^2 + \sigma \eta_t \]

**AR(1)** (Held & Kleinen’04)  
DFA (Livina & Lenton’07)  

**Variance**

\[ \text{AR(1)} \ x(t_{n+1}) = \alpha x(t_n) \]

Variance  

stationary distribution non-normal  

Guttal  
Livina, Kwasniok, Lenton’10  

generalisation poor
Estimates for nonlinear parts

**Fokker-Planck equation**

Density \( p \) of

\[
\dot{x} = f(x, \mu) + \sigma \eta_t
\]

does not satisfy

\[
\frac{\partial_t p}{\partial x} = \frac{\sigma^2}{2} \partial_{xx} p - \partial_x [f(x, \mu) p]
\]

the stationary density \( p(x) \)

\[
\frac{1}{2} \partial_x p(x) = \sigma^{-2} f(x, \mu) p(x) + c
\]
Estimates for nonlinear parts

**Fokker-Planck equation**

Density $p$ of $\dot{x} = f(x, \mu) + \sigma \eta_t$

satisfies

$$\partial_t p = \frac{\sigma^2}{2} \partial_{xx} p - \partial_x [f(x, \mu) p]$$

Stationary density $p(x)$

$$\frac{1}{2} \partial_x p(x) = \sigma^{-2} f(x, \mu) p(x) + c$$

empirical
Estimates for nonlinear parts

**Fokker-Planck equation**  
Density \( p \) of \( \dot{x} = f(x, \mu) + \sigma \eta_t \) satisfies

\[
\partial_t p = \frac{\sigma^2}{2} \partial_{xx} p - \partial_x [f(x, \mu) p]
\]

Stationary density \( p(x) \)

Linear estimate \(-\kappa x\)

\[
\frac{1}{2} \partial_x p(x) = \sigma^{-2} f(x, \mu) p(x) + \text{empirical}
\]
Paleo-climate records

End of last glaciation

$\text{time} = -1.7 \times 10^4$

End of Younger Dryas

$\text{time} = -11.6 \times 10^3$

Temperature greyscale

Potential well

2.8% 10.8%

Fitted c

Empirical skewness

18% 41%

Fitted c

Empirical skewness
Summary

- accuracy of estimates
  zero order > first order > second order
- but second order term necessary to estimate tipping time/probability
- estimate tipping time/probability based on saddle-node normal form

[JMTT,JS on arxiv]