Entropy and Prime Number Distribution; (a Non-heuristic Approach).

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> "Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate."
> - L. Euler (1770).

The distribution of the prime numbers is not indeterminate but it is nevertheless disorderly[1]. This is a consequence of the presence of entropy determining the placement of the primes in the continuum of real numbers.

Hitherto the presence of entropy has merely been intuited as a consequence of the apparent randomness of prime number distribution and therefore we have been able to measure its presence in only the most heuristic of fashions.[2]

This is ultimately unsatisfactory since what is required is a law like approach which enables us to determine once and for all that the fundamental reason for the apparent randomness of the distribution of prime numbers is entropy. This would then enable us to understand not only that the disorderly distribution of the primes is an inherent and fundamental feature of the continuum of real numbers but will also shed light on the paradoxical regularities (or analytical properties) that seem to underlie this distribution pattern, regularities that are strongly implied by the Prime Number Theorem and the

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[1] It has been drawn to my attention that "disorder" is not currently recognized as a precise technical term in mathematics. This is a consequence of the heuristic treatment of entropy in number theory which this paper aims to address. In physics by contrast entropy has a precise physical significance and is used synonymously with entropy. This is due to the work of Ludwig Boltzmann who successfully reinterpreted thermodynamics as a theory of order (and disorder). In consequence, measuring the entropy of a system is equivalent to measuring its disorder. It is in the light of this reinterpretation of thermodynamics that I use the term *disorder* throughout this paper.

[2] For an example of the heuristic approach to the measurement of entropy using probabilistics, entropy series etc. see for example;

R. Granville; *Harald Cramer and the Distribution of Prime Numbers; Scandinavian Actuarial J*. 1 (1995), 12-28). Web reference;

http://www.maths.ex.ac.uk/~mwatkins/zeta/physics6.htm#cramer

S.W. Golans; *Probability, Information Theory and Prime Number Theory.* Discrete Mathematics 106-7 (1992) 219-229.

C. Bonnano and M.S. Mega; *Toward a Dynamical Model for Prime Numbers.* Chaos, Solitons and Fractals 20. (2004) 107-118.

The above two papers and many on similar topics may be accessed at;
http://www.maths.ex.ac.uk/~mwatkins/zeta/NTentropy.htm

An alternative to the heuristic approach (or perhaps a variant of it) is the suggestion that conjectures of sufficient intractability *and* generality (in practice the Riemann hypothesis and the P versus NP problem are the most commonly cited examples) should be accorded (by fiat) the status of intuitive mathematical axioms. Doubtless the continuum hypothesis would also have been included in this list had it not been solved by now. Arguments for this idea together with an account of its history are to be found in;

G.J. Chaitin; *Thoughts on the Riemann Hypothesis; arXiv:math.HO/0306042 v4 22 Sep 2003.* Web reference;
http://www.cs.auckland.ac.nz/CDMTCS/chaitin

In view of what I will have to say on the Riemann hypothesis and the P versus NP problem it is obvious that I do not agree with what seems to me to be an unduly pessimistic approach. Furthermore, neither problem seems to me to be sufficiently elementary or fundamental enough to be considered axiomatic anyway.

Instead what is required is an expansion of the field of mathematics so as to create a new perspective on these two intimately connected problems. It is my contention that this expansion is made possible by means of a *non-heuristic* treatment of the problem of entropy in mathematics.

Riemann Hypothesis. These analytical properties, it turns out, are no more than the paradoxical calling card of entropy.

What is required then is a law-like equation allowing us to precisely measure the exact amount of entropy in the continuum of real numbers at any given point. It is the primary purpose of this paper to supply this equation, its complete derivation and also something of its profound significance.

The key to deriving this equation lies in exploiting an hitherto unnoticed link that exists between Ludwig Boltzmann's well known equation for entropy and the historically contemporaneous Prime Number Theorem of Jacques Hadamard and Charles de la Vallee Poussin. This connection lies in the fact that both formalisms make use of the natural logarithm $\log_e(x)$.

In the case of Boltzmann's statistical interpretation of entropy the natural logarithm amounts to a measurement of entropy – i.e. of disorder or randomness.

$$\frac{s}{k} = \log_e(x) \tag{1}$$

That is; as $\log_e(x)$ increases disorder S also increases, with k being Boltzmann's constant. The value for Boltzmann's constant is approximately 1.38^{-23} Joules/Kelvin.

It is valid to recalibrate the value for entropy so as to extinguish any direct reference to Boltzmann's constant, effectively making our measurement of entropy dimensionless[3]. This has the substantial benefit of corresponding exactly to Shannon's Information Entropy and may be represented (for example) as

$$S\hat{}=\ln x \tag{2}$$

Boltzmann's use of the natural logarithm is obviously different to that of the Prime Number Theorem. Boltzmann intended his equation to be used to measure the disorderliness of thermodynamical systems. Accordingly the value for x in $\log_e(x)$ is intended to represent the number of potential microstates that a given thermodynamical system could possibly inhabit. Consequently, the larger the thermodynamical system is the larger will be the value for x and hence (ipso-facto) the larger will be its entropy.

In the Prime Number Theorem however the x in $\log_e(x)$ refers not to the microstates of a thermodynamical system but rather to particular positive integers;

$$\pi(x) \approx \frac{x}{\log_e x} \approx Li(x) \tag{3}$$

My proposal (which will allow us to blend the above two formalisms) is that the continuum of real numbers be treated as a *geometrical* system and that the positive integers be then interpreted as representing possible microstates within that system, thereby allowing us to measure (in a dimensionless way) the relative magnitude of entropy represented by any given positive integer. For example; the number one will represent one microstate, the number seventy eight will represent seventy eight microstates, the number two thousand and sixty two will represent two thousand and sixty two microstates and so on.

[3] I shall represent this recalibration by means of the sign S^.

This transposition is legitimate because Boltzmann's equation is famous for treating thermodynamical systems as if they are geometrical systems. In the case of the continuum of real numbers the shift (from thermodynamics to geometry) is unnecessary since the continuum of real numbers is manifestly a geometrical system anyway.

If this rationale is granted then it allows us to blend the above two formalisms (Boltzmann's equation for entropy and the Prime Number Theorem). This may occur in the following fashion;

$$\log_e x \approx \frac{x}{\pi(x)} \approx \frac{s}{k} \tag{4}$$

Which simplifies to give us;

$$S \approx k \frac{x}{\pi(x)} \tag{5}$$

Recalibrating this to make it dimensionless so as to correspond with Shannon's information entropy we are left with the following result;

$$S\wedge \approx \frac{x}{\pi(x)} \tag{6}$$

In essence the entropy of any positive integer is equal to the integer itself divided by $\pi(x)$ (i.e. divided by the number of primes up to x). [4]

Consequently, when we plug positive integers into the above equation as values for x what we find is a statistical tendency for entropy to increase as values for x get larger – which accords with what we would intuitively expect to find.

Rather bizarrely however this tendency is merely statistical in nature and sometimes leads to larger entropy readings for smaller positive integers and vice versa. For example the number eleven has a smaller entropy reading than the number ten. That is, there is less entropy associated with the number eleven then there is with the number ten.

This sort of paradoxical effect, (in which entropy temporarily goes into reverse, becoming negative rather than positive), recurs throughout the continuum, albeit with a diminishing frequency.

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[4] Note also this highly suggestive definition of an integer x;

$$x \approx \frac{s}{k}.\pi(x)$$

which recalibrates to;

$$x \approx s\wedge.\pi(x)$$

i.e. an integer (x) is defined by its entropy multiplied by $\pi(x)$ (i.e. multiplied by the number of primes up to x).
This equation may point to the very essence of what number is.

What is truly remarkable is that these stochastic entropy reversal phenomena are invariably associated with the appearance of new primes in the continuum. Whenever a new prime appears entropy temporarily goes into recession. It is as though the primes give the continuum an infusion of order (or energy).

What this means in practice is that in order to enable entropy to increase, the distributional density of the primes must decline in a progressive fashion, much as is predicted by the Prime Number Theorem. If the distributional density of the primes did not diminish then entropy could not increase in the continuum. And this fact effectively explains the otherwise mysterious regularity in the distribution of primes implied by the Prime Number Theorem. This regularity (which is famously given visual form in the smooth rise of the graph of $\pi(n)$,[5]) far from contradicting the presence of entropy in the distribution pattern of the primes, is in reality a vital prerequisite allowing positive entropy to manifest itself in the continuum.

This entropy reversal phenomenon may seem absurd to the scientist or mathematician trained in classical physics or classical mathematics but it is a common feature of non-classical physics where entropy is well known to reverse itself, albeit only ever fleetingly, at the quantum scale. In consequence of this we have the phenomenon of quantum indeterminacy and the uncertainty principle. But the first inkling of this non-classical effect dates back to Boltzmann's statistical equation.

Since entropy can be known independently of experience it is fair to say that it is more fundamental than the merely empirical phenomenon it was hitherto assumed to be. It has, in effect, been elevated to the status of a logical category.

Incidentally, we should mention at this point that the equally disorderly distribution of non-trivial zeros outputted by the Riemann Zeta-Function indicates the presence of entropy in the continuum of imaginary numbers as well. To be more precise; the non-trivial complex zeros outputted by the Riemann Zeta Function are *(for their imaginary part alone)* distributed in a disorderly fashion. Obviously the real parts of these zeros are distributed in an orderly way (if the Riemann hypothesis is true) an orderliness which is nothing more than the shadow cast by entropy.

It follows then that the so called "critical line" is a mirror (on the imaginary plane) for the "prime line" (on the real plane), since the disorderly distribution of the primes is mirrored by that of the imaginary part of Riemann's zeros. This effectively retains the strict symmetry between the two continua. It also indicates the consistent presence of entropy in both continua, possibly for the first time.

Entropy is present in both continua simply because it is present in everything that exists, including metaphysical or logical objects such as numbers. The reason why this is so is because entropy is a pre-requisite for the existence of phenomena, a fact first identified in Curie's Principle[6]. It appears therefore that entropy is a pre-requisite for the existence of logical objects (such as continua) as well. Phenomena (including the continua of real and imaginary numbers) cannot come into being without the shattering of symmetry and the shattering of symmetry is the function performed by entropy.

And entropy may also account for some of the many interesting links detected between physics, cosmology and the Riemann Zeta-function. They are similar effects with entropy as their common cause. Since entropy is the same dynamical phenomenon in physics as it is in prime number distribution it should not be surprising that these similarities occur.

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[5] This mysterious graph is, so to speak, the shadow cast by entropy.

[6] This principle states that all phenomena are the product of symmetry violations. It is first given in;
Pierre Curie*; Sur la symmetrie dans les phenomenes physiques*. Journal de physique. 3rd series. Volume 3. p 393-417.

Proof of the Riemann Hypothesis;

The Riemann hypothesis can itself be interpreted as a conjecture about entropy. This follows because of Helge von Koch's well known version of the Riemann hypothesis which shows the hypothesis to be simply a stronger version of the prime number theorem;

$$\pi(x) = Li_{(x)} + 0(x^{\frac{1}{2}} \log x) \tag{7}$$

Bearing in mind (6) and (7) it follows that;

$$S = k.\left(\frac{x}{Li_{(x)} + 0(x^{\frac{1}{2}} \log x)}\right) = k.\frac{x}{\pi_{(x)}} \tag{8}$$

And this new equation too may be recalibrated so as to make it dimensionless;

$$S\,\hat{}= \frac{x}{Li_{(x)} + 0(x^{\frac{1}{2}} \log x)} \tag{9}$$

(7) may also be expressed as;

$$\pi(x) = \int_2^t \frac{dt}{\log t} + 0(\sqrt{x} \log x) \tag{10}$$

from which it follows that;

$$S = k.\frac{x}{\int_2^t \frac{dt}{\log t} + 0(\sqrt{x}} \frac{}{\log x} \tag{11}$$

Which again can be recalibrated as;

$$S\hat{} = \frac{x}{\int_2^t \frac{dt}{\log t} + 0(\sqrt{x}} \frac{}{\log x} \tag{12}$$

It is partly due to von Koch's work that we know just how intimately the Riemann hypothesis is connected to the issue of prime number distribution. And this all important connection was further emphasized to me in a private communication by Professor Marcus du Sautoy;

> "The Riemann hypothesis is equivalent to proving that the error between Gauss's guess and the real number of primes up to N is never more than the square root of N – the error that one expects from a random process."[7]

Now unlike Professor du Sautoy I am no true expert on the Riemann hypothesis but (bearing in mind equations (7) to (12)) it seems to me that if his observations are true (and it is admittedly unfair to place too much weight on merely informal comments) then it should logically follow that proving the disorderly (as distinct from indeterminate) nature of prime number distribution should amount (ipso-facto) to a proof of the Riemann hypothesis. This is because the error mentioned (the square root of N) is the error that a disorderly distribution of the primes necessarily gives rise to. Since equation (6) demonstrates to us that the primes are indeed *intrinsically* disorderly in their distribution it therefore follows (from (6)) that the Riemann hypothesis must be true. To put it another way; demonstrating the central role of entropy in determining the distribution of the primes has the incidental effect (if Professor du Sautoy's analysis is correct) of confirming the Riemann hypothesis.

Proof of the P versus NP Problem[8];

(6) also seems to cast its incidental light on the problem of factorization[9]. It clearly implies that the efficient factorization of very large numbers (of the order of magnitude used to construct R.S.A. encryption codes for example) cannot be achieved in deterministic polynomial time.

This is because if such an efficient factorization algorithm existed it would immediately contradict what we now know from (6) concerning the intrinsically disordered nature of prime number distribution. Thus if the factorization problem were in P (i.e. were subject to a "shortcut" algorithmic solution) it would contradict (6). One can either have disorderly primes *or* one can have an efficient factorization algorithm, *one cannot logically have both.*

If the factorization problem were in P it would mean that prime number distribution is orderly. Therefore, the factorization problem, though in NP cannot logically be in P. Ergo;

$$P \neq NP$$

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[7] Professor du Sautoy has elsewhere written that;
"So if Riemann was correct about the location of the zeros, then the error between Gauss's guess for the number of primes less than N and the true number of primes is at most of the order of the square root of N. This is the error margin expected by the theory of probability if the coin is fair, behaving randomly with no bias…
… To prove that the primes are truly random, one has to prove that on the other side of Riemann's looking-glass the zeros are ordered along his critical line." – M. du Sautoy; *The Music of the Primes.* Harper Collins (2003). P167.
The *converse* of this crucial information is that in order to prove the Riemann hypothesis it is sufficient to prove the truly random (or, more precisely, the *disorderly)* distribution of the primes. This function is clearly performed by equation (6).
[8] For a fairly full introduction to this problem see web reference;
S. Cook; *The P Versus NP Problem; http://www.claymath.org/millennium/P_vs_NP/Official_Problem_Description.pdf*
[9] Given an integer *n* try to find the prime numbers which, when multiplied together, give *n*.

Thus it is not necessary to prove that the factorization problem is NP complete for it to serve as a valid counter-example to P and NP equivalence. Proving that the factorization problem is NP complete is only necessary as part of the process of proving P and NP equivalence.
Though not NP complete the factorization problem is nevertheless known to be in NP since it is a problem whose solutions can at least be checked in P time.

Proof of the Primality of the Number One;

(6) is a dimensionless equation, nevertheless the results it outputs should be measured in calibrated units with a particular base. When we plug the number one into (6) it outputs an infinite answer. Although we know that entropy in the continuum tends towards infinity as the continuum tends towards infinity we also know that the number one cannot be associated with infinite entropy.
This absurd result is simply a product of (mis)treating the number one as if it were not a prime. This results in $\pi(1)$ being zero, therefore leading to an infinite value for entropy.
However, if the number one is correctly treated as a prime number it results in $\pi(1)$ being one, thus leading to an entropy value (in effect a base entropy value) of one. This leads us to conclude that the number one, contrary to historical practice, must logically be treated as a prime number in its own right.[10] Indeed, since it supplies us with our base value for entropy it should be regarded as the cardinal prime.
Just as water gives us our base for measuring physical density (calibrated at one unit) so, even more naturalistically, the number one supplies us with our unit base for veridically measuring entropy in the continuum.
A consequence of this unavoidable reinterpretation is that the value of $\pi(x)$ must shift by plus one for every integer in the continuum. Thus, for example, $\pi(10)$ becomes 5 (leading to an entropy reading of 2 units) and $\pi(11)$ becomes 6 (as does $\pi(12)$), generating entropy readings of 1.83^{r} and 2 units respectively. $\pi(13)$ and $\pi(14)$ meanwhile become 7, leading to entropy ratings of approximately 1.857142 and 2 units. Consequently the entropy reading for the number ten is equal to or higher than it is for the numbers eleven through to fourteen.[11]
This example therefore admirably illustrates our earlier thesis concerning the entropy reversal phenomena triggered by the appearance of new primes. Consequently if the density of primes does not decline (as indicated by the prime number theorem) then the steady statistical advance of entropy in the continuum will be halted. This then is obviously the function of the effect (of the thinning of the primes) first detected by Gauss (i.e. to facilitate the expansion of entropy in the continuum).
Another phenomenon worthy of note at this juncture is that of *entropy assonance* whereby the entropy of different integers is sometimes identical. For example, the integers ten, twelve and fourteen all produce readings of two units of entropy and the first three positive integers of the continuum all share entropy readings of one unit. This kind of assonance can only occur as a by-product of the appearance of a new prime number.

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[10] The arguments against primality, in contrast, are weak and informal, metaphysical even.

[11] These figures raise the question of what the graph of the function $f(x) = \dfrac{x}{\pi(x)}$ looks like (bearing in mind my proposition that one is prime).

Proof of the Twin Primes Conjecture;

It also seems to me that the disorderly distribution of prime numbers confirms certain k-tuplet conjectures such as the twin primes conjecture. This is because if there were not an infinite number of twin primes then this state of affairs would in turn imply the existence of some sort of "hidden order" in prime number distribution preventing their accidental recurrence. Since we now know that prime numbers are inherently disorderly in their distribution it therefore follows that this cannot be the case. Ergo there must be an infinite number of twin primes.

This reasoning should also prove transferable to a treatment of the Mersenne primes conjecture.

Incidentally, what might be called the "entropy differential" between twin primes is usually very small and diminishes at an exponential rate. For example, the entropy differential between the twin primes three and five is 25% (because the entropy readings of these numbers are one and one and a quarter units respectively). However, for the twin primes eleven and thirteen the differential has already shrunk to barely more than 1% (with entropy readings of 1.83r units and 1.857 units respectively). Although this rate of shrinkage continues throughout the continuum the differential itself only completely vanishes at infinity.

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References;

[1] R. Granville; *Harald Cramer and the Distribution of Prime Numbers; Scandinavian Actuarial J*. 1 (1995), (12-28).
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