Zeros of the Zeta Function

The best references for the theory of the Riemann zeta function are:

H. Edwards, *Riemann's zeta function*, Academic Press (1974).

A. Ingham, *The distribution of prime numbers*, Cambridge University Press, Cambridge, 1932.

S. Patterson, *An introduction to the theory of the Riemann zeta-function*, Cambridge, Cambridge-New York, 1988.

E. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, edited by D. R. Heath-Brown, Oxford University Press, 1986.

I will follow Ingham for the proof of the key bound (5), which he attributes to Backlund.

Since the gamma function has no zeros, and since the Riemann zeta function has an Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

which shows it to be nonvanishing in the right half plane $\mathrm{re}(s) > 1$, the function $\Lambda(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$ has no zeros in $\mathrm{re}(s) > 1$. By the functional equation $\Lambda(s) = \Lambda(1-s)$, it also has no zeros in $\mathrm{re}(s) < 0$. Thus all the zeros have their real parts between 0 and 1.

Conjecture (Riemann). *If* $\Lambda(s) = 0$, *then* $\mathrm{re}(s) = \frac{1}{2}$.

This is the *Riemann hypothesis.* Over a billion zeros of the zeta function have been calculated, beginning with $\frac{1}{2} + i14.134725\ldots$, and it has been possible to verify the Riemann hypothesis does not fail in this range.

Lemma. $\Lambda(s) = \Lambda(1-s) = \overline{\Lambda(\overline{s})}$. *The function* Λ *is real on the line* $\mathrm{re}(s) = 1/2$.

Proof. The identity $\Lambda(s) = \Lambda(1-s)$ is the functional equation. The identity $\Lambda(s) = \overline{\Lambda(\overline{s})}$ follows from the fact that Λ is real on the real line, so $\Lambda(s) - \overline{\Lambda(\overline{s})}$ is an analytic function vanishing on the real line, hence zero since the zeros of an analytic function which is not identically zero can have no accumulation point. We note that if $s = \frac{1}{2} + it$ where t is real, then $\overline{s}$ and $1-s$ coincide, so this implies that $\Lambda(s)$ is real. $\square$

This Lemma shows that we may detect zeros of $\zeta(s)$ on the line $\mathrm{re}(s) = 1/2$ by detecting sign changes in $\Lambda(\frac{1}{2} + it)$, so there is no need to compute the location of a zero *exactly* in order to confirm that it is on this line. Thus if we compute

$$\Lambda(\tfrac{1}{2} + 14i) = -2.05141 \times 10^{-6}, \qquad \Lambda(\tfrac{1}{2} + 15i) = 6.26591 \times 10^{-6},$$

we know (without computing it exactly) that there is a zero of $\zeta(\frac{1}{2}+it)$ with t between 14 and 15.

Let T be a positive real number, and let $N(T)$ be the number of zeros of $\zeta(s)$ in the region $0 \le \mathrm{re}(s) \le 1$, $0 \le \mathrm{im}(s) \le T$. It is possible to compute $N(T)$ exactly, for large T, by a method which we shall discuss below. It is then possible to check the Riemann hypothesis numerically by computing $N(T)$ for some fixed T, then exhibiting $N(T)$ sign changes in $\Lambda(\frac{1}{2}+it)$ between $t=0$ and $t=T$. Of course it is impossible to prove the Riemann hypothesis this way, but this type of confirmation has been helpful in giving confidence in the conjecture.

We recall that if f is a meromorphic function in a simply connected domain Ω, and γ is a Jordan curve in Ω traversed counterclockwise which avoids the zeros and poles of f, then it is a consequence of the residue theorem that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(s)}{f(s)}\,ds$$

is equal to the number of zeros, minus the number of poles of f. (Indeed, it is easy to see from the Taylor expansion that if a is a zero of order n, or pole of order $-n$, then n is the residue of f'/f at a.)

We may write

$$\frac{f'(s)}{f(s)}\,ds = d\,\log\big(f(s)\big).$$

Of course $\log f(s)$ is only defined up to addition of a constant of $2\pi i$, but locally near some point on the path, it is possible to assign values of log to make a continuous function.

Sometimes it is convenient to note that since this value is guaranteed to be an integer, hence real, we may as well take the imaginary part of the integral before dividing by $2\pi i$, and write the number of zeros (minus poles) as

$$\frac{1}{2\pi}\mathrm{im}\int_\gamma \frac{f'(s)}{f(s)}\,ds.$$

In this formulation, this fact is sometimes referred to as *the principle of the argument*, for reasons we will now explain. If $z = re^{i\theta}$ is a complex number, $\theta = \arg(z)$ is the *argument*. Like $\log(z)$, it is only defined up to a constant. We have

$$\mathrm{im}\,\frac{f'(s)}{f(s)}\,ds = d\arg\big(f(s)\big),$$

which we may see as follows. It is sufficient to check that $\mathrm{im}\,\log\big(f(s)\big) = \arg\big(f(s)\big)$, and indeed $\mathrm{im}\,\log(re^{i\theta}) = \mathrm{im}(\log(r)+i\theta) = \theta$. Thus the number of zeros (minus poles) of

f inside γ is equal to $1/2\pi$ times the amount by which $\arg\big(f(s)\big)$ is augmented when s traverses the path γ—the principle of the argument.

We consider the integral

$$\frac{1}{2\pi i}\int \frac{\Lambda'(s)}{\Lambda(s)}\,ds \tag{1}$$

around the rectangle with vertices at $2-iT$, $2+iT$, $-1+iT$ and $-1-iT$. This is equal to the number of zeros, minus the number of poles of $\Lambda(s)$ inside the rectangle. There are two poles, at $s=0$ and $s=1$; and there are $2\,N(T)$ zeros, since the zeros above the real axis are mirrored below. Thus the integral (1) equals $2\,N(T)-2$.

The path of integration can be broken into four pieces,

(i) 2 to $2+iT$ to $\frac{1}{2}+iT$;
(ii) $\frac{1}{2}+iT$ to $-1+iT$ to -1;
(iii) -1 to $-1-iT$ to $-\frac{1}{2}-iT$;
(iv) $-\frac{1}{2}-iT$ to $2-iT$ to 2.

Each of these integrals contributes equally to (1), because of the relations

$$\Lambda(s)=\Lambda(1-s)=\overline{\Lambda(\overline{s})}=\overline{\Lambda(1-\overline{s})}.$$

Consequently,

$$\frac{1}{\pi i}\int_2^{\frac{1}{2}+iT}\frac{\Lambda'(s)}{\Lambda(s)}\,ds=N(T)-1, \tag{2}$$

where the path of integration is taken over the contour (i), vertically to $2+it$, then horizontally to $\frac{1}{2}+it$. Now we exploit the additivity of the logarithmic derivative. We have

$$\frac{\Lambda'(s)}{\Lambda(s)}\,ds=d\log\,\Lambda(s)=d\log\Big(\pi^{-s/2}\,\Gamma\big(\tfrac{s}{2}\big)\Big)+d\log\,\zeta(s).$$

Now $\pi^{-s/2}\,\Gamma\big(\frac{s}{2}\big)$ is analytic and nonzero in the region $\mathrm{re}(s)>0$, so its logarithm can be defined as an analytic function in this region, and

$$\int_2^{\frac{1}{2}+iT} d\log\Big(\pi^{-s/2}\,\Gamma\big(\tfrac{s}{2}\big)\Big)=\log\Big(\pi^{-\frac{1}{4}-\frac{iT}{2}}\,\Gamma\left(\tfrac{1}{4}+\tfrac{iT}{2}\right)\Big)-\log\big(\pi^{-1}\,\Gamma(1)\big).$$

Also, since (2) is guaranteed to be real (in fact, an integer), we are justified in taking imaginary parts before dividing by πi, and

$$N(T)-1=\frac{1}{\pi}\,\mathrm{im}\left[\log\Big(\pi^{-\frac{1}{4}-\frac{iT}{2}}\,\Gamma\left(\tfrac{1}{4}+\tfrac{iT}{2}\right)\Big)-\log\big(\pi^{-1}\,\Gamma(1)\big)+\int_2^{\frac{1}{2}+iT}\frac{\zeta'(s)}{\zeta(s)}\,ds\right].$$

Taking imaginary parts this way will simplify the bookkeeping. We will make use of the "big O" notation, and denote any function $F(T)$ such that $|F(T)/f(T)|$ is bounded as $O\big(f(T)\big)$. Thus (by Stirling's formula) we write

$$\log\Gamma\big(\tfrac{1}{4}+\tfrac{IT}{2}\big) = \big(\tfrac{iT}{2}-\tfrac{1}{4}\big)\log\big(\tfrac{iT}{2}\big) - \tfrac{iT}{2} + O(1), \tag{3}$$

meaning that

$$\log\Gamma\big(\tfrac{1}{4}+\tfrac{IT}{2}\big) - \big(\tfrac{iT}{2}-\tfrac{1}{4}\big)\log\big(\tfrac{iT}{2}\big) - \tfrac{iT}{2}$$

is bounded. Thus

$$N(T) = \tfrac{1}{\pi}\,\mathrm{im}\left[\big(-\tfrac{1}{4}-\tfrac{iT}{2}\big)\log(\pi) + \big(\tfrac{iT}{2}-\tfrac{1}{4}\big)\log\big(\tfrac{iT}{2}\big) - \tfrac{iT}{2} + \int_2^{\frac{1}{2}+iT}\frac{\zeta'(s)}{\zeta(s)}\,ds\right] + O(1),$$

or since

$$\log\big(\tfrac{iT}{2}\big) = \log\big(\tfrac{T}{2}\big) + \tfrac{i\pi}{2},$$

we obtain

$$N(T) = \tfrac{T}{2\pi}\log\big(\tfrac{T}{2\pi}\big) - \tfrac{T}{2\pi} + \tfrac{1}{\pi}\,\mathrm{im}\int_2^{\frac{1}{2}+iT}\frac{\zeta'(s)}{\zeta(s)}\,ds + O(1).$$

Now we note that since ζ is holomorphic and nonvanishing in the half-plane $\mathrm{re}(s) > 1$, because the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

in which each factor is nonzero, is convergent in this region. If t is real,

$$\int_2^{2+it}\frac{\zeta'(s)}{\zeta(s)}\,ds = \log\zeta(2+it) - \log\zeta(2).$$

Here

$$|\zeta(2+it)-1| = |\sum_{n=2}^{\infty} n^{-2+it}| \le \sum_{n=2}^{\infty}|n^{-s}| = \zeta(2)-1 = .644934\cdots.$$

Since this is less than 1, $\zeta(2+it)$ is constrained to a circle which excludes the origin, and

$$|\zeta(2+it)| > 1 - .644934\cdots. \tag{4}$$

Thus

$$\int_2^{2+it}\frac{\zeta'(s)}{\zeta(s)}\,ds = O(1),$$

and we may write

$$N(T) = \tfrac{T}{2\pi}\log\left(\tfrac{T}{2\pi}\right) - \tfrac{T}{2\pi} + \tfrac{1}{\pi}\,\mathrm{im}\int_{2+iT}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)}\,ds + O(1). \tag{5}$$

We will prove below that

$$\mathrm{im}\int_{2+iT}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)}\,ds = O(\log T). \tag{6}$$

Therefore we will obtain

$$N(T) = \tfrac{T}{2\pi}\log\left(\tfrac{T}{2\pi}\right) - \tfrac{T}{2\pi} + O(\log T).$$

First we need to know a bound for ζ on vertical strips. Let $s = \sigma + it$ where σ and t are real. (This notation is traditional since the paper of Riemann.)

Proposition 1. *Let $0 < \delta < 1$. In the region $\sigma \geq \delta$, $t > 1$, we have*

$$\zeta(\sigma + it) = O(t^{1-\delta}).$$

A better estimate can be obtained by a standard argument using the functional equation and the Phragmén-Lindelöf principle. This standard argument gives $O(t^{\epsilon+(1-\delta)/2})$ for any $\epsilon > 0$. Further improvements have been a topic of active investigation, the crucial issue being the growth on the line $\mathrm{re}(s) = \frac{1}{2}$. According to the *Lindelöf hypothesis*, $\zeta(\frac{1}{2} + it) = O(t^{\epsilon})$ for any $\epsilon > 0$. This would follow from the Riemann hypothesis.

Proof. We will deduce the meromorphic continuation of $\zeta(s)$ to the region $\mathrm{re}(s) > 0$ by Euler-Maclaurin summation, and the formula we get will give the estimate of the Proposition. Let N be a large integer to be determined later. If f is any smooth function, for $M > N$ we recall that

$$\sum_{n=N}^{M} f(n) = \int_N^M f(x)\,dx + \tfrac{1}{2}f(N) + \tfrac{1}{2}f(M) + \int_N^M B_1(x - [x])\,f'(x)\,dx,$$

where $B_1(x) = x - \frac{1}{2}$ is the first Bernoulli polynomial, and $[x]$ is the greatest integer. Take $f(x) = x^{-s}$, where initially $\mathrm{re}(s) > 1$, and let $M \to \infty$. We obtain

$$\begin{aligned}\zeta(s) - \sum_{n<N} n^{-s} = \sum_{n=N}^{\infty} n^{-s} &= \int_N^\infty x^{-s}\,dx + \tfrac{1}{2}N^{-s} - s\int_N^\infty x^{-s-1}\,B_1(x - [x])\,dx \\ &= \frac{N^{1-s}}{1-s} + \tfrac{1}{2}N^{-s} - s\int_N^\infty x^{-s-1}\,B_1(x - [x])\,dx.\end{aligned}$$

This relation, originally defined if $\mathrm{re}(s) > 0$ has analytic continuation to the left, for the integral

$$s \int_N^\infty x^{-s-1}\, B_1(x - [x])\, dx$$

is absolutely convergent if $\sigma = \mathrm{re}(s) > 0$, and since $|B_1(x - [x])| < \frac{1}{2}$,

$$\left| s \int_N^\infty x^{-s-1}\, B_1(x - [x])\, dx \right| < \frac{|s|}{2} \int_N^\infty x^{-\sigma-1}\, dx = \frac{|s|}{2\sigma} N^{-\sigma} \le \left(\frac{1}{2} + \frac{t}{2\sigma} \right) N^{-\sigma},$$

where we have used the triangle inequality $|s| \le \sigma + t$. Also

$$\left| \sum_{n<N} n^{-s} \right| \le \sum_{n<N} n^{-\sigma} < \int_0^N x^{-\sigma}\, dx = \frac{N^{1-\sigma}}{1-\sigma},$$

and

$$\left| \frac{N^{1-s}}{1-s} \right| \le \frac{N^{1-\sigma}}{t}.$$

Thus

$$\begin{aligned} |\zeta(s)| &= \left| \sum_{n<N} n^{-s} + \frac{N^{1-s}}{1-s} + \tfrac{1}{2} N^{-s} - s \int_N^\infty x^{-s-1}\, B_1(x - [x])\, dx \right| \\ &\le \frac{N^{1-\sigma}}{1-\sigma} + \frac{N^{1-\sigma}}{t} + \left(\frac{1}{2} + \frac{t}{2\sigma} \right) N^{-\sigma}. \end{aligned}$$

Assuming that $t > 1$, we may estimate this by taking N to be the greatest integer less that t.

Note: To see that this is the optimal choice of t, consider the two potentially largest terms, $N^{1-\sigma}/(1-\sigma)$ and $(t/2\sigma)\, N^{-\sigma}$. If we take N to be approximately t^α for some α, these are $t^{\alpha(1-\sigma)}/(1-\sigma)$ and $(2\sigma)^{-1}\, t^{1-\alpha\sigma}$. As α varies, one increases, the other decreases. Thus, we want to equate the exponents, so $\alpha(1-\sigma) = 1 - \alpha\sigma$, or $\alpha = 1$.

Taking $N \sim t$, we see that $\zeta(s)$ is of the order $O(t^{1-\sigma})$. If $\sigma \ge \delta$, and $t > 1$, we see that $\zeta(\sigma + it) = O(t^{1-\delta})$. ■

Proposition 2. *Let f be a function which is analytic in a neighborhood of the disk $|z - a| < R$. Suppose that $0 < r < R$, and that f has n zeros in the disk $|z - a| < r$. Let $M = \max |f(a + Re^{i\theta})|$, and suppose that $|f(0)| \ne 0$. Then*

$$\left(\frac{R}{r} \right)^n \le \frac{M}{|f(0)|}.$$

PROOF. This is a typical application of Jensen's formula, which asserts that if $z_1, \cdots, z_m$ are the zeros of f inside $|z-a| < R$, then

$$\sum \log \left|\frac{R}{z_i}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\ |f(a+Re^{i\theta})| - \log|f(0)|.$$

The sum on the left is greater than or equalt to the contribution from those z_i inside $|z-a| < r$, which in turn is greater than or equal to $n\ \log|R/r|$. Thus

$$n\ \log\left|\frac{R}{r}\right| \le \log|M| - \log|f(0)|,$$

and exponentiating, we obtain the Proposition. ■

Theorem. *We have*

$$N(T) = \tfrac{T}{2\pi}\log\left(\tfrac{T}{2\pi}\right) - \tfrac{T}{2\pi} + O(\log T).$$

PROOF. In view of (5), it is sufficient to prove the estimate (6). We may assume that the path from $2+iT$ to $\frac{1}{2}+iT$ does not pass through any zero of $\zeta(s)$, by moving the path up slightly if necessary. By the principle of the argument, the integral

$$\operatorname{im}\int_{2+iT}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)}\,ds$$

represents the change in the argument of $\zeta(s)$ as s moves from $2+iT$ to $\frac{1}{2}+iT$. This is approximately πc, where c is the number of sign changes in $\operatorname{re}\zeta(s+iT)$ as s moves from 2 to $\frac{1}{2}$, since the sign must change every time the argument changes by π. We note that if s is real, $\operatorname{re}\zeta(s+iT) = \frac{1}{2}\big(\zeta(s+iT)+\zeta(s-iT)\big)$. Therefore, it is sufficient to show that the number of zeros of $\frac{1}{2}\big(\zeta(s+iT)+\zeta(s-iT)\big)$ on the segment $[\frac{1}{2}, 2]$ of the real axis is $O(\log\ T)$. In fact, we will use Proposition 2 to estimate the number of zeros of $f(s) = \frac{1}{2}\big(\zeta(s+iT)+\zeta(s-iT)\big)$ inside the circle $|s-2| < \frac{3}{2}$. We take $a=2$, $R=7/4$ and $r=3/2$ in the Proposition.

First, we note that $|f(2)|$ is bounded below by (4). On the other hand,

$$\max_{|s-2|=7/4} |f(s)| = O(T^{3/4})$$

by Proposition 1. Therefore if n is the number of zeros of f inside $|s-2| < \frac{3}{2}$, we have

$$\left(\frac{7/4}{3/2}\right)^n = O(T^{3/4}),$$

or taking logarithms, $n\ \log(7/6)$ is bounded by $\frac{3}{4}\ \log(T)$ plus a constant. This completes the proof. ■